Tools

MA310

Spring 2001

1 Winning Positions

We have considered some two-person games. A typical game has a certain (usually finite) number of *positions* from each one of which either player has a certain (usually finite) number of *moves* to other positions.

If either player has the same choices of moves from each position, then the game is called *impartial*. So Tic-Tac-Toe is *not* impartial, because different players must use different symbols, but the game in which players take turns removing one or more stones out of one of three piles of stones *is* impartial.

One way to try to analyze an impartial game is to identify each position as either *winning* or *losing*. The "goal" position(s) of the game is, of course, winning. By "working backwards" from the goal position(s), we can identify other positions as winning or losing by the following requirements:

1. If all moves from a given position P lead to losing positions, then P is a winning position.

2. If there is at least one move from a given position P that leads to a winning position, then P is a losing position.

If the total number of possible positions in the game is finite, and if there are no sequences of moves that allow you to return to a previous position in the game, then the above method can in principle identify the winning positions. But certain games may be quite complicated, and we may instead want to figure out and prove some sort of *criterion* to identify the winning positions.

2 Symmetry

Some of our games have winning positions described by symmetry. Points A and B are *symmetrical with respect to a point* P if P is the midpoint of the line segment joining A and B. Two points A and B are *symmetrical with respect to a line* ℓ if the line ℓ is the perpendicular bisector of the line segment joining A and B.

3 Remainders

Given integer a and positive integer b, there are integers q and r such that $a = qb + r$ and $0 \leq r < b$. q is called the *quotient* and r the *remainder*. In this case we can say that a *is congruent to* r *mod* b. It is not hard to prove that if a has remainder r when divided by b, then so does $a + b$, and so does a plus any integer multiple of b.

4 Magic Squares

An $n \times n$ *magic square* is an arrangement of numbers, usually the integers from 1 to n^2, in an $n \times n$ array such that the sum of the numbers in each row, column, and diagonal is the same. Except for rotating or reflecting it, there is only one 3×3 magic square:

8	1	6
3	5	7
4	9	2