Reasoning and Proof

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena. People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove. Ultimately, a mathematical proof is a formal way of expressing particular kinds of reasoning and justification.

Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and—with different expectations of sophistication—at all grade levels, students should see and expect that mathematics makes sense. Building on the considerable reasoning skills that children bring to school, teachers can help students learn what mathematical reasoning entails. By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments.

Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in geometry. Proof is a very difficult area for undergraduate mathematics students. Perhaps students at the postsecondary level find proof so difficult because their only experience in writing proofs has been in a high school geometry course, so they have a limited perspective (Moore 1994). Reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts.

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Recognize reasoning and proof as fundamental aspects of mathematics

From children's earliest experiences with mathematics, it is important to help them understand that assertions should always have reasons. Questions such as "Why do you think it is true?" and "Does anyone think the answer is different, and why do you think so?" help students see that statements need to be supported or refuted by evidence. Young children may wish to appeal to others as sources for their reasons ("My sister told me so") or even to vote to determine the best explanation, but students need to learn and agree on what is acceptable as an adequate argument in the *mathematics* classroom. These are the first steps toward realizing that mathematical reasoning is based on specific assumptions and rules.

Part of the beauty of mathematics is that when interesting things happen, it is usually for good reason. Mathematics students should understand this. Consider, for example, the following "magic trick" one might find in a book of mathematical recreations:

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> Write down your age. Add 5. Multiply the number you just got by 2. Add 10 to this number. Multiply this number by 5. Tell me the result. I can tell you your age. »

The procedure given to find the answer is, Drop the final zero from the number you are given and subtract 10.

The result is the person's age. Why does it work? Students at all grade levels can explore and explain problems such as this one.

Systematic reasoning is a defining feature of mathematics. It is found in all content areas and, with different requirements of rigor, at all grade levels. For example, first graders can note that even and odd numbers alternate; third graders can conjecture and justify—informally, perhaps, by paper folding—that the diagonals of a square are perpendicular. Middle-grades students can determine the likelihood of an even or odd product when two number cubes are rolled and the numbers that come up are multiplied. And high school students could be asked to consider what happens to a correlation coefficient under linear transformation of the variables.

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Make and investigate mathematical conjectures

Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery. Teachers and researchers agree that students can learn to make, refine, and test conjectures in elementary school. Beginning in the earliest years, teachers can help students learn to make conjectures by asking questions: What do you think will happen next? What is the pattern? Is this true always? Sometimes? Simple shifts in how tasks are posed can help students learn to conjecture. Instead of saying, "Show that the mean of a set of data doubles when all the values in the data set are doubled," a teacher might ask, "Suppose all the values of a sample are doubled. What change, if any, is there in the mean of the sample? Why?" High school students using dynamic geometry software could be asked to make observations about the figure formed by joining the midpoints of successive sides of a parallelogram and attempt to prove them. To make conjectures, students need multiple opportunities and rich, engaging contexts for learning.

Young children will express their conjectures and describe their thinking in their own words and often explore them using concrete materials and examples. Students at all grade levels should learn to investigate their conjectures using concrete materials, calculators and other tools, and increasingly through the grades, mathematical representations and symbols. They also need to learn to work with other students to formulate and explore their conjectures and to listen to and understand conjectures and explanations offered by classmates.

p. 57 Teachers can help students revisit conjectures that hold in one context to check to see whether they still hold in a new setting. For instance, the common notion that "multiplication makes bigger" is quite appropriate for young children working with whole numbers larger than 1. As they move to fractions, this conjecture needs to be revisited. Students may not always have the mathematical knowledge and tools they need to find a justification for a conjecture or a counterexample to refute it. For example, on the basis of their work with graphing calculators, high school students might be quite convinced that if a polynomial function has a value that is greater than 0 and a value that is less than 0 then it will cross the *x*-axis somewhere. Teachers can point out that a rigorous proof requires more knowledge than most high school students have. »

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Develop and evaluate mathematical arguments and proofs

Along with making and investigating conjectures, students should learn to answer the question, Why does this work? Children in the lower grades will tend to justify general claims using specific cases. For instance, students might represent the odd number 9 as in figure 3.5 and note that "an odd number is something that has one number left over" (Ball and Bass forthcoming, p. 33). Students might then reason that any odd number will have an "extra" unit in it, and so when two odd numbers are added, the two "extra" units will become a pair, giving an even number, with no "extras." By the upper elementary grades, justifications should be more general and can draw on other mathematical results. Using the fact that congruent shapes have equal area, a fifth grader might claim that a particular triangle and rectangle have the same area because each was formed by dividing one of two congruent rectangles in half. In high school, students should be expected to construct relatively complex chains of reasoning and provide mathematical reasons. To help students develop and justify more-general conjectures and also to refute conjectures, teachers can ask, "Does this always work? Sometimes? Never? Why?" This extension to general cases draws on more-sophisticated mathematical knowledge that should build up over the grades.

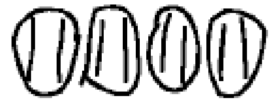

Fig. **3.5.** A representation of 9 as an odd number

Students can learn about reasoning through class discussion of claims that other students make. The statement, If a number is divisible by 6 and by 4, then it is divisible by 24, could be examined in various ways. Middle-grades students could find a counterexample—the number 12 is divisible by 6 and by 4 but not by 24. High school students might find a related conjecture involving prime numbers that they could verify. Or students could explore the converse. In any event, both plausible and flawed arguments that are offered by students create an opportunity for discussion. As students move through the grades, they should compare their ideas with others' ideas, which may cause them to modify, consolidate, or strengthen their arguments or reasoning. Classrooms in which students are encouraged to present their thinking and in which everyone contributes by evaluating one another's thinking provide rich environments for learning mathematical reasoning.

Young children's explanations will be in their own language and often will be represented verbally or with objects. Students can learn to articulate their reasoning by presenting their thinking to their groups, their classmates, and to others outside the classroom. High school students should be able to present mathematical arguments in written forms that would be acceptable to professional mathematicians. The particular format of a mathematical justification or proof, be it narrative argument, "two-column proof," or a visual argument, is less important than a clear and correct communication of mathematical ideas appropriate to the students' grade level.

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Select and use various types of reasoning and methods of proof

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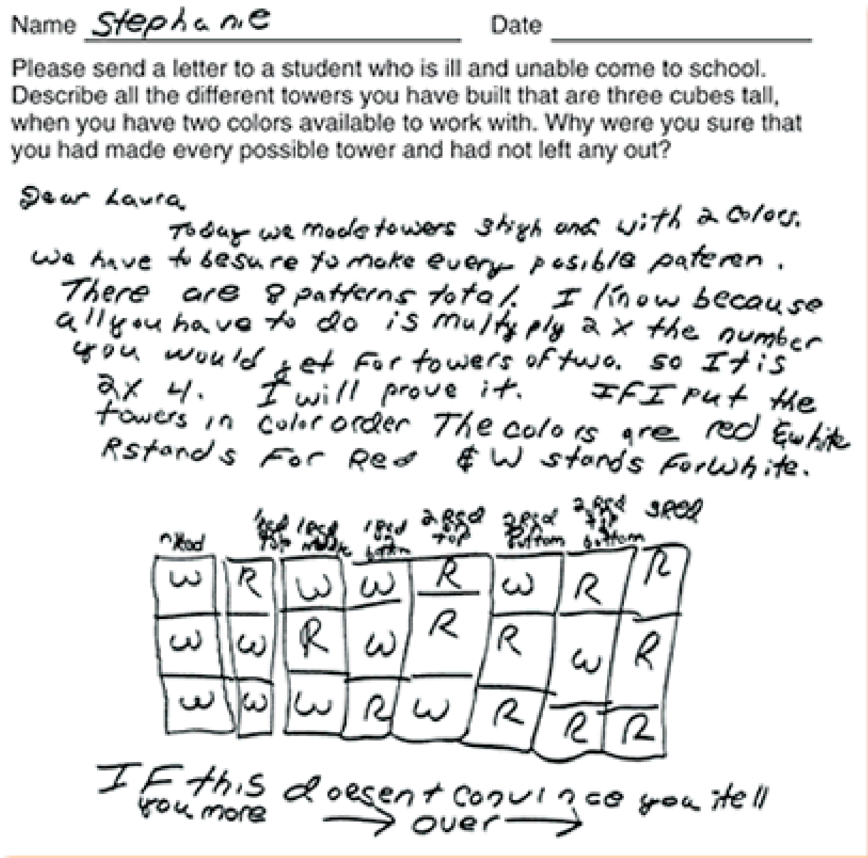
In the lower grades, the reasoning that children learn and use in mathematics class is informal compared to the logical deduction used » by the mathematician. Over the years of schooling, as teachers help students learn the norms for mathematical justification and proof, the repertoire of the types of reasoning available to students—algebraic and geometric reasoning, proportional reasoning, probabilistic reasoning, statistical reasoning, and so forth—should expand. Students need to encounter and build proficiency in all these forms with increasing sophistication as they move through the curriculum.

Young children should be encouraged to reason from what they know. A child who solves the problem 6 + 7 by calculating 6 + 6 and then adding 1 is drawing on her knowledge of adding pairs, of adding 1, and of associativity. Students can be taught how to make explicit the knowledge they are using as they create arguments and justifications.

Early efforts at justification by young children will involve trial-and-error strategies or the unsystematic trying of many cases. With guidance and many opportunities to explore, students can learn by the upper elementary grades how to be systematic in their explorations, to know that they have tried all cases, and to create arguments using cases. One research study (Maher and Martino 1996, p. 195) reported a fifth grader's elegant proof by cases in response to the problem in figure 3.6.


Name Stephanie Date

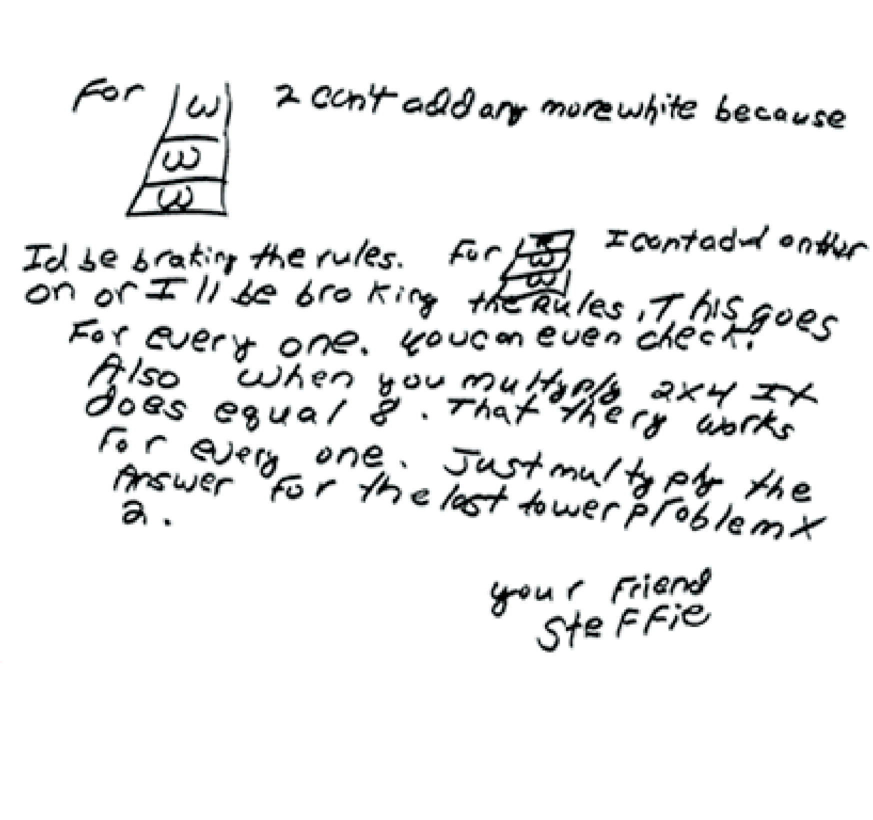
Please send a letter to a student who is ill and unable come to school. Describe all the different towers you have built that are three cubes tall, when you have two colors available to work with. Why were you sure that you had made every possible tower and had not left any out?

Dear Laura,

Today we made towers 3 high and with 2 colors. We have to be sure to make every posible pateren. There are 8 patterns total. I know because all you have to do is multiply 2 x the number you would get for towers of two. so It is 2x 4. I will prove it. If I put the towers in color order The colors are red & white R stands for Red & W stands for White.

no Red	1 Red top	1 Red middle	1 Red bottom	2 Red top	2 Red bottom	2 Red top bottom	3 Red
W	R	W	W	R	W	R	R
W	W	R	W	R	R	W	R
W	W	W	R	W	R	R	R

If this doesent convince you tell you more → over →


For [W/W/W tower] I can't add any more white because I'd be braking the rules. For [tower] I can't add another on or I'll be braking the Rules, This goes For every one. you can even check! Also when you multiply 2x4 It does equal 8. That thery works for every one. Just multyply the answer for the last tower problem x 2.

your Friend
Steffie

Fig. **3.6.** Stephanie's elegant "proof by cases" produced in grade 5 (from Maher and Martino [1996])

Proof by contradiction is also possible with young children. A first grader argued from his knowledge of whole-number patterns that the number 0 is even: "If 0 were odd, then 0 and 1 would be two odd numbers in a row. Even and odd numbers alternate. So 0 must be even." Beginning in the elementary grades, children can learn to disprove conjectures by finding counterexamples. At all levels, students will reason inductively from patterns and specific cases. Increasingly over the grades, they should also learn to make effective deductive arguments based on the mathematical truths they are establishing in class.