Problems

1 Problems I

1.1 Soccer Game

At the conclusion of a soccer game whose two teams each included 11 players, each player on the winning team "gave five" to (slapped hands with) each player on the losing team. Each player on the winning team also gave five to each *other* player on the winning team. How many fives were given? —*Crossing the River with Dogs*, p. 7.

1.2 Elevator

The capacity of an elevator is either 20 children or 15 adults. If 12 children are currently in the elevator, how many adults can still get in? —*Crossing the River with Dogs*, p. 7.

1.3 Theater Group

There are eight more women than men in a theater group. The group has a total of 44 members. How many men and how many women are in the group? —*Crossing the River with Dogs*, p. 8.

1.4 Ducks and Cows

Farmer Brown had ducks and cows. One day she noticed that the animals had a total of 12 heads and 32 feet. How many of the animals were ducks and how many were cows? —*Crossing the River with Dogs*, p. 8.

1.5 Strange Number

If you take a particular two-digit number, reverse its **digits** to make a second two-digit number, and add these two numbers together, their sum will be 121. What is the original number? —*Crossing the River with Dogs*, p. 8.

1.6 Counterfeit Coins

Suppose you have a collection of nine coins, eight of which are of equal weight, and the ninth which is counterfeit and a little bit heavier.

1. How can you use a two-pan balance scale to perform weighings of various piles of the coins to determine which coin is counterfeit? How many weighings are required to guarantee finding the counterfeit coin?

2. Now suppose you have twelve coins, among which is one counterfeit coin, which may be either a little bit lighter or a little bit heavier. How many weighings will you need to find the counterfeit coin?

—*National Library of Virtual Manipulatives*, Coin Problem.

2 Problems II

2.1 Chickens and Eggs

If a chicken and a half can lay an egg and a half in a day and a half, how long would it take two chickens to lay a dozen eggs?

2.2 Ones and Twos

There are five ways to express the number 4 in terms of adding together 1's and 2's: $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, and $2+2$.

1. In how many ways can you express the number 10?
2. In how many ways can you express the positive integer n?

2.3 Handshaking

At a recent party someone decided to keep track of how many other people each person shook hands with. It was not necessarily the case that everyone shook hands with everyone else. Nevertheless, it turned out that the number of individuals who shook hands an *odd* number of times was *even*. Explain why this must be so.

2.4 Polygon Clusters

Find all the different ways you can select three regular polygons of equal side length (but not necessarily with the same number of sides) that will fit together snugly around a common point. One way, for example, is to use three regular hexagons.

2.5 Sum to One Half

Find all positive integers a, b, c that will solve the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$.

2.6 Bookshelf

1. You have a box of 5 different books but room on your bookshelf for only 2 of them. How many ways can you select and order a set of 2 books?
2. What if you have room for all 5?

3. What if you have 11 books in the box but room for 6 of them?
4. What if you have m books in the box, but room for n of them?
5. What if you have m books in the box, and have room for all of them?

2.7 Packing Books

You are going on a relaxing vacation and want to bring along some reading material. There are 10 books you recently purchased that you haven't read yet.

1. If you only had room to pack 2 books, how many choices would you have of pairs of books to bring?
2. What if you had room for 4 books? How many choices would you have of collections of 4 books to bring?
3. What if you had m books and could take along only n of them?
4. What if you had 20 books and could take any number of them (including all or none)?

2.8 Counting Pairings

1. How many ways can 5 different men be completely paired up with 5 different women? For example, if we labeled the men $M_1, \ldots, M_5$ and the women $W_1, \ldots, W_5$, then one complete pairing could be $\{M_1W_1, M_2W_2, M_3W_3, M_4W_4, M_5W_5\}$ and a second complete pairing could be $\{M_1W_2, M_2W_3, M_3W_1, M_4W_4, M_5W_5\}$.
2. How many ways can n different men be completely paired up with n different women?

3 Problems III

3.1 How Many Questions?

You are going to play a game with a friend. Your friend is going to think of a whole number in the range 0 to 127, inclusive. You are going to ask "yes" or "no" questions to determine the number. What is the fewest number of questions that you can ask to eventually determine the number?

3.2 Prime Numbers

Find all the prime numbers between 1 and 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

3.3 Number Guessing Cards

Here are two versions of the Number Guessing Cards. One person chooses a number that appears on at least one card, then indicates on which cards the chosen number appears. The other person then must as quickly as possible determine which number was chosen.

Small Set

CARD 1	**CARD 2**	**CARD 3**
1	2	4
3	3	5
5	6	6
7	7	7

Large Set

CARD 1	CARD 2	CARD 3	CARD 4	CARD 5
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

3.4 Binary Numbers

When we express numbers in the base 10 system, we use the symbols 0 through 9 as digits. The places of the digits are associated with powers of 10. The rightmost digit is in the 1's place, and moving left we then have the 10's place, the 100's place, the 1000's place, and so on. For example, the numerals 4253 represent the number 4 thousands, 2 hundreds, 0 tens, and 3 ones.

When we express numbers in the base 2 (or binary) system, we use the symbols 0 and 1 as digits. The places of the digits are associated with powers of 2. The rightmost digit is in the 1's place, and moving left we then have the 2's place, the 4's place, the 8's place, and so on. For example, the numerals 1101 represent the number 1 eight, 1 four, 0 twos, and 1 one (which is the number 13 when written in base ten).

1. Make a table expressing each of the whole numbers from 0 to 31 in base 2.

n	16	8	4	2	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	0
3	0	0	0	1	1
⋮	⋮	⋮	⋮	⋮	⋮

2. Study this table and then explain how the Number Guessing Cards work.

3.5 Weird Multiplication I

Why does the following method work?

Suppose you want to calculate 18×14. Write these two numbers at the top of two columns. In the first column, start with 18, divide it by two, and repeat, ignoring any remainders, until you reach the number 1. In the second column, start with 14, double it, and repeat, until you have a column of equal length. Draw a line through any number in the first column that is even, and extend the line to cross out the neighboring number in the second column. Add up the remaining numbers in the second column. This will be the product, and this procedure works for any pair of numbers. Two examples:

$$\begin{array}{rr} \cancel{18} & \cancel{14} \\ 9 & 28 \\ \cancel{4} & \cancel{56} \\ \cancel{2} & \cancel{112} \\ 1 & 224 \\ \hline & 252 \end{array} \qquad \begin{array}{rr} 13 & 25 \\ \cancel{6} & \cancel{50} \\ 3 & 100 \\ 1 & 200 \\ \hline & 325 \end{array}$$

3.6 Weird Multiplication II

Why does the following method work?

Number the fingers on each hand 5, 6, 7, 8, and 9, starting with your pinkies and ending with your thumbs. Hold your hands out in front of you with the palms facing you, thumbs up, and pinkies down.

To multiply two numbers on the above list together, touch the corresponding fingers together. For example, to calculate 8×7, touch the index finger of your left hand to the middle finger of your right hand.

At and above the index finger on your left hand there are 2 fingers. At and above the middle finger on your right hand there are 3 fingers. Multiply these two numbers together: $2 \times 3 = 6$. The remaining fingers on both hands combined each count 10, and there are 5 of them, for a product of 50. Add 6 and 50 to get the answer: $8 \times 7 = 56$.

Confused? Let's try another one: 6×8. Touch the ring finger of your left hand to the index finger of your right hand. Multiplying the number of fingers at and above the touching fingers: $4 \times 2 = 8$. Counting fingers below the touching fingers as tens: 40. Add 8 and 40 to get the answer: 48.

3.7 Watching TV

In a suburban home live Abner, his wife Beryl and their three children, Cleo, Dale and Ellsworth. The time is 8 p.m. on a winter evening.

1. If Abner is watching television, so is his wife.
2. Either Dale or Ellsworth, or both of them, are watching television.
3. Either Beryl or Cleo, but not both, is watching television.
4. Dale and Cleo are either both watching or both not watching television.
5. If Ellsworth is watching television, then Abner and Dale are also watching.

Who is watching television and who is not?

3.8 Five Houses

There are five houses, each a different color and inhabited by men of different nationalities with different pets, drinks, and favorite foods.

1. The Englishman lives in the red house.
2. The Spaniard owns the dog.
3. Coffee is drunk in the green house.
4. The Ukrainian drinks tea.
5. The green house is immediately to your right of the white house.
6. The pizza eater owns snails.

7. Sushi is eaten in the yellow house.
8. Milk is drunk in the middle house.
9. The Norwegian lives in the first house on the left.
10. The man who eats burritos lives next to the man with the fox.
11. Sushi is eaten next to the house with a horse.
12. The steak eater drinks orange juice.
13. The Japanese eats jambalaya.
14. The Norwegian lives next to the blue house.

Who drinks water and who owns the zebra?

4 Problems IV

4.1 100

This is a two-player game. The first player selects a number between 1 and 6, inclusive. To this number the second player adds a number from 1 through 6. The players continue to alternately increase the current sum, each by adding a number from 1 through 6. The first player to achieve the exact sum 100 wins. What is a good strategy? Who will win?

4.2 Penny Piles

This is another two-player game. The initial position is three piles of pennies (or some other object). The first pile has 3 pennies, the second pile has 5 pennies, and the third has 7 pennies. Players alternately take turns, which consists of removing any positive number of pennies from any one pile. The winner is the player who removes the very last penny. What is a good strategy? Who will win?

4.3 Three Towers

The Three Towers puzzle consists of three posts and n disks of different sizes that are initially stacked on one of the posts, from largest to smallest with the largest disk on the bottom. The goal is to move this stack of disks to a different post by a sequence of moves. Each move consists of moving the disk from the top of the pile on any post to the top of the pile on any other post, as long as a larger disk is never placed directly on a smaller one. Try to solve this puzzle with $n = 7$. Can you find a general procedure for n disks? How many moves are required?

4.4 Fifteen

This is a two-player game. There is a deck of 9 cards consisting of the ace through the 9 of clubs displayed on the table. Players alternately select a card from the currently unchosen cards. The first player to acquire, among his/her cards, a set of three cards that sum to 15, wins. (An ace counts as 1.) If no player is able to make a set after all 9 cards have been selected, the game is a tie. What is a good strategy? Who can win?

4.5 School Lockers

A new school is being opened. The school has exactly 1000 lockers and 1000 students. On the first day of school, the students meet outside the building and agree on the following plan: The first student will enter the school and open all of the lockers. The second student will enter and close every locker with an even number. The third student will then "reverse" every third locker beginning with the third locker; that is, if the locker is closed, he will open it; if it's open, he will close it. The fourth student will then "reverse" every fourth locker beginning with the fourth locker; and so on until all 1000 students in turn have entered the building and "reversed" the proper lockers. Which lockers will finally remain open?

5 Problems V

5.1 Folding Cubes

A *pentomino* is a flat figure made by attaching five squares together along some of their edges. For example, you could simply make one long chain of five squares, or you could make two types of L's.

A *hexomino* is a flat figure made by attaching six squares together along some of their edges.

1. Make drawings of all the different pentominoes. (If one can be gotten from another by rotations or reflections, then don't count them as different.) How can you be systematic so that you *know* that you have them all?

2. How many of these can be folded up into a cubical box with an open top?

3. Make drawings of all the different hexominoes. (If one can be gotten from another by rotations or reflections, then don't count them as different.) How can you be systematic so that you *know* that you have them all?

4. How many of these can be folded up into a closed cubical box?

5.2 Regular Polyhedra

A convex polyhedron is *regular* if all of its faces are identical regular polygons, and the same number of faces meet at each vertex. Find them all. How can you be systematic so that you *know* that you have them all? Note: It is not permitted that two adjacent faces be coplanar. Note also that "convex" means that for every possible line segment with both endpoints on the surface of the polyhedron, the rest of the line segment is in the interior; i.e., the polyhedron has no "dimples" or indentations.

5.3 Semiregular Polyhedra

A convex polyhedron is *semiregular* if all of its faces are regular polygons, there are at least two different types of faces, and the same configuration (type and sequence) of polygons meets at each vertex. For example, you can make a polyhedron out of equilateral triangles and squares such that the configuration "triangle-square-triangle-square" (or 3-4-3-4 for short) *in that order* meets at each vertex.

1. Find all the semiregular polyhedra that can be constructed with various combinations of equilateral triangles, squares, regular pentagons, and regular hexagons. Note: It is not permitted that two adjacent faces be coplanar.

2. Use outside sources to find drawings and the names of all of the semiregular polyhedra.

5.4 Deltahedra

A convex polyhedron is a *deltahedron* if it is made entirely of equilateral triangles. Find them all. Note: It is not permitted that two adjacent faces be coplanar.

5.5 Counting Faces

For each of the various polyhedra that you have found, count the numbers of vertices, edges, and faces, and conjecture a relationship that holds among these three numbers.