Suppose that \( n \) is a positive integer. A partition of \( n \) is a way of writing \( n \) as a sum of positive integers, where we order the summands from least to greatest. For example, three partitions of 7 are

\[
7 = 1 + 3 + 3 \\
7 = 1 + 2 + 2 + 2 \\
7 = 3 + 4.
\]

We do not consider sums like

\[
7 = 2 + 1 + 2 + 2
\]

because we want the terms in the sum to increase as we read them from left to right.

One thing mathematicians have been very interested in over time are partition identities, where a partition identity is a statement that the number of partitions of one kind are equal to the number of partitions of another kind. We will spend today investigating some lovely partition identities.

1. The number of partitions of \( n \) into \( m \) parts equals the number of partitions of \( n \) whose greatest part is \( m \).

2. The number of partitions of \( n \) into at most \( m \) parts of size at most \( k \) equals the number of partitions of \( n \) into at most \( k \) parts of size at most \( m \).

3. The number of partitions of \( n \) that are self conjugate equals the number of partitions of \( n \) into distinct odd parts.

4. (This one is due to Euler) The number of partitions of \( n \) into odd parts equals the number of partitions of \( n \) into distinct parts.

And here are two problems on counting partitions:

8. How many partitions are there with at most \( m \) parts of size at most \( k \)? Use this setup to prove the recurrence for binomial coefficients (“Pascal’s triangle”).

9. Let \( c_m \) denote the number of partitions whose \( k \)th part is at most \( k - 1 \) (so their Ferrer’s diagram fits into a “staircase”). Show that

\[
c_{m + 1} = c_0 c_m + c_1 c_{m - 1} + \cdots + c_m c_0,
\]

where we define \( c_0 = 1 \). The numbers \( c_m \) are called Catalan numbers and are given by the formula

\[
c_m = \frac{1}{m + 1} \binom{2m}{m}.
\]