Algebra Examples

Some things to remember:

- "And" usually corresponds to intersection.
- "Or" usuall corresponds to union.
- The square root of a positive number is by convention the positive square root.
- $\sqrt{A^2} = |A|$.
- Pay close attention to the validity, reversibility, and nonreversibility of logical implications.

1. Solve $x^2 = 25$.

   (a) Valid solution.

$$\begin{array}{c} x^2 = 25 \\ \Updownarrow \\ \sqrt{x^2} = \sqrt{25} \\ \Updownarrow \\ |x| = 5 \\ \Updownarrow \\ (x \geq 0 \text{ and } x = 5) \text{ or } (x \leq 0 \text{ and } -x = 5) \\ \Updownarrow \\ (x = 5) \text{ or } (x = -5) \end{array}$$

   The solution set is $\{-5, 5\}$.

   (b) Valid solution.

$$\begin{array}{c} x^2 = 25 \\ \Updownarrow \\ x^2 - 25 = 0 \\ \Updownarrow \\ (x+5)(x-5) = 0 \\ \Updownarrow \\ (x+5 = 0) \text{ or } (x-5 = 0) \\ \Updownarrow \\ (x = -5) \text{ or } (x = 5) \end{array}$$

   The solution set is $\{-5, 5\}$.

(c) Solution method to avoid.

$$x^2 = 25$$
$$\Updownarrow$$
$$x = \pm 5 \text{ by taking the square root of both sides}$$

The solution set is $\{-5, 5\}$.

[Even though the correct answer is achieved, we are taking both roots on the right hand side, which can lead to problems in other problems if we are not careful—see the next problem, for example.]

2. Solve $x^2 < 4$.

(a) Valid solution.

$$x^2 < 4$$
$$\Updownarrow$$
$$\sqrt{x^2} < 2$$
$$\Updownarrow$$
$$|x| < 2$$
$$\Updownarrow$$
$$(x \geq 0 \text{ and } x < 2) \text{ or } (x \leq 0 \text{ and } -x < 2)$$
$$\Updownarrow$$
$$(0 \leq x < 2) \text{ or } (-2 < x \leq 0)$$
$$\Updownarrow$$
$$-2 < x < 2$$

The solution set is $\{x : -2 < x < 2\}$.

(b) Valid solution.

$$\begin{gathered} x^2 < 4 \\ \Updownarrow \\ x^2 - 4 < 0 \\ \Updownarrow \\ (x+2)(x-2) < 0 \\ \Updownarrow \\ (x+2 > 0 \text{ and } x-2 < 0) \text{ or } (x+2 < 0 \text{ and } x-2 > 0) \\ \Updownarrow \\ (x > -2 \text{ and } x < 2) \text{ or } (x < -2 \text{ and } x > 2) \\ \Updownarrow \\ (-2 < x < 2) \text{ or } (x \in \emptyset) \\ \Updownarrow \\ -2 < x < 2 \end{gathered}$$

The solution set is $\{x : -2 < x < 2\}$.

(c) Invalid Solution.

$$\begin{gathered} x^2 < 4 \\ \Updownarrow \\ \sqrt{x^2} < 2 \\ \Updownarrow \\ x < 2 \end{gathered}$$

The solution set is $\{x : x < 2\}$.

[The Solution is invalid because although $\sqrt{x^2} < 2$ implies $x < 2$, the converse is not true. Indeed, it is possible to have $x < 2$ but not have $\sqrt{x^2} < 2$.]

(d) Invalid Solution.

$$\begin{gathered} x^2 < 4 \\ \Updownarrow \\ \sqrt{x^2} < \pm 2 \\ \Updownarrow \\ x < \pm 2 \end{gathered}$$

The solution set is $\{x : x < \pm 2\}$.

[What does this statement mean? $x < 2$ and $x < -2$? $x < 2$ or $x < -2$? Neither are correct solutions. And if we try $\pm x < \pm 2$, then we generate $x < 2$, $-x < 2$, $x < -2$, and $-x < -2$. It is not clear what to do with all of these in a consistent manner.]

3. Solve $x^2 > 5$.

   (a) Valid solution.

$$\begin{array}{c} x^2 > 5 \\ \Updownarrow \\ \sqrt{x^2} > \sqrt{5} \\ \Updownarrow \\ |x| > \sqrt{5} \\ \Updownarrow \\ (x \geq 0 \text{ and } x > \sqrt{5}) \text{ or } (x \leq 0 \text{ and } -x > \sqrt{5}) \\ \Updownarrow \\ (x > \sqrt{5}) \text{ or } (x < -\sqrt{5}) \end{array}$$

   The solution set is $\{x : (x > \sqrt{5}) \text{ or } (x < -\sqrt{5})\}$.

   (b) Valid solution.

$$\begin{array}{c} x^2 > 5 \\ \Updownarrow \\ x^2 - 5 > 0 \\ \Updownarrow \\ (x + \sqrt{5})(x - \sqrt{5}) > 0 \\ \Updownarrow \\ (x + \sqrt{5} > 0 \text{ and } x - \sqrt{5} > 0) \text{ or } (x + \sqrt{5} < 0 \text{ and } x - \sqrt{5} < 0) \\ \Updownarrow \\ (x > -\sqrt{5} \text{ and } x > \sqrt{5}) \text{ or } (x < -\sqrt{5} \text{ and } x < \sqrt{5}) \\ \Updownarrow \\ (x > \sqrt{5}) \text{ or } (x < -\sqrt{5}) \end{array}$$

   The solution set is $\{x : (x > \sqrt{5}) \text{ or } (x < -\sqrt{5})\}$.

   (c) Invalid solution.

$$\begin{array}{c} x^2 > 5 \\ \Updownarrow \\ \sqrt{x^2} > \sqrt{5} \\ \Updownarrow \\ x > \sqrt{5} \end{array}$$

   The solution set is $\{x : x > \sqrt{5}\}$.

[The solution is invalid because $\sqrt{x^2} > \sqrt{5}$ does not imply $x > \sqrt{5}$ unless you know in advance that $x \geq 0$.]

(d) Invalid solution.

$$\begin{array}{c} x^2 > 5 \\ \Updownarrow \\ \sqrt{x^2} > \sqrt{5} \\ \Updownarrow \\ x > \pm\sqrt{5} \end{array}$$

The solution set is $\{x : x > \pm\sqrt{5}\}$.

[What does this statement mean? $x > \sqrt{5}$ and $x > -\sqrt{5}$? $x > \sqrt{5}$ or $x > -\sqrt{5}$? Neither are correct solutions. And if we try $\pm x > \pm\sqrt{5}$, then we generate $x > \sqrt{5}$, $-x > \sqrt{5}$, $x > -\sqrt{5}$, and $-x > -\sqrt{5}$. It is not clear what to do with all of these in a consistent manner.]