Games that are Numbers

1. Some, but not all, partizan games are numbers!
2. A *dyadic* number is a rational number that of the form $\frac{a}{b}$ where a is an integer and b is a power of 2: $b \in \{1, 2, 4, 8, \ldots\}$.
3. Consider an interval (a, b), in which $a = -\infty$ and $b = \infty$ are allowed. The *simplest* or *oldest* number in the interval is defined as follows.
   (a) If the interval contains at least one integer, the simplest number in the interval is the integer with smallest absolute value.
   (b) If the interval contains no integer, the simplest number in the interval is the (unique) dyadic number that can be expressed with the smallest power of two in its denominator.
4. We consider partizan games in normal form with finite game trees. (But note that this theory can be extended to certain partizan games with infinite game trees.)
   (a) The position $\{ \mid \}$ has the value 0.
   (b) A position $\gamma = \{\alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_m\}$ can be assigned a value if both of the following conditions hold:
      i. Each α_i and each β_j has an assigned value a_i and b_j, respectively.
      ii. There is no pair a_i, b_j such that $a_i \geq b_j$.

      In this case γ is assigned the dyadic value c, where c is the simplest number greater than every a_i and less than every b_j.
5. If α and β are two games with values, then $\alpha \equiv \beta$ if and only if their values are the same.
6. The value of the sum of games is the sum of their values.
7. Some examples of partizan games that are numbers (have values): Checker Stacks (see below), Cut-Cake, Hackenbush.
8. An example of a partizan game that is not a number (does not have values for every position): Domineering.

9. In order for L to win, L must always move to a position that is non-negative. In order for R to win, R must always move to a position that is nonpositive. Recommended strategy for L: Make the move that reduces the value of the game the least. Recommended strategy for R: Make the move that increases the value of the game the least.

10. Checker Stacks (a special case of Hackenbush with a single strand.)

    (a) The value of a stack of n L checkers is n. This game is called $\bullet n$.

    (b) The value of a stack of n R checkers is $-n$. This game is called $\bullet - n$ or $- \bullet n$.

    (c) The value of the stack

$$\begin{array}{c} R \\ \vdots \\ R \\ L \end{array}$$

    in which there are n R's is $\frac{1}{2^n}$. This game is called $\bullet\frac{1}{2^n}$.

    (d) Short-cut for any single pile of Checker Stacks for which the bottom checker is L. Begin with the value 0. Starting from the bottom of the pile and moving upward, add 1 for each L. When you reach the first R, subtract $\frac{1}{2}$. The next checker is worth $+\frac{1}{4}$ if L and $-\frac{1}{4}$ if R. The next checker is worth $+\frac{1}{8}$ if L and $-\frac{1}{8}$ if R. The next checker is worth $+\frac{1}{16}$ if L and $-\frac{1}{16}$ if R. Etc.

    Use the same process with opposite signs if the bottom checker is R.