MA 341 — Homework #1

Solutions

We saw in class that an equation of a line containing the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$(y_1 - y_2)x + (x_2 - x_1)y = x_2y_1 - x_1y_2.$$

Knowing this, now assume that (x_1, y_1), (x_2, y_2), and (x_3, y_3) are three distinct points and consider the expression

$$A = x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2.$$

1. Prove that if the three points all lie on a common line then $A = 0$.

   Solution: Let's call the three points P_1, P_2, and P_3, respectively. We know that the line determined by P_1 and P_2 is given by the equation $(y_1 - y_2)x + (x_2 - x_1)y = x_2y_1 - x_1y_2$. If P_3 is collinear with these two points, then its coordinates (x_3, y_3) will satisfy the equation of the line. So make this substitution and rearrange:

   $$\begin{aligned} (y_1 - y_2)x_3 + (x_2 - x_1)y_3 &= x_2y_1 - x_1y_2 \\ x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2 &= 0 \end{aligned}$$

   Therefore $A = 0$.

2. Conversely, prove that if $A = 0$ then the three points all lie on a common line.

   Solution: Assume that $A = 0$. Then rearranging we have:

   $$\begin{aligned} x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2 &= 0 \\ (y_1 - y_2)x_3 + (x_2 - x_1)y_3 &= x_2y_1 - x_1y_2 \end{aligned}$$

   So the point P_3 satisfies the equation of the line determined by P_1 and P_2. Therefore the three points are collinear.

3. Do a lot of experiments with actual points to try to figure out what geometric meaning the quantity A has. That is to say, if you plot the three points, make a good guess backed by your evidence as to what A measuring. It is not necessary (yet) to prove your conjecture.

   Solution: A plausible conjecture is that the area of the triangle determined by the three points is $|A/2|$. Using GeoGebra to make lots of examples is very helpful.