3 Coordinates

3.1 The Analytical Euclidean Plane $\mathbf{E}^2$

3.1.1 Equations of Lines

Suppose you have two distinct points (x_1, y_1) and (x_2, y_2) in $\mathbf{E}^2$. You probably already know several ways of getting an equation for the line containing them. For example, if $x_1 \neq x_2$, you can calculate the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and use the *point-slope* formula

$$y - y_1 = m(x - x_1).$$

Here is a variation on this formula that might be new to you, and works even if the line turns out to be vertical:

An equation of the line containing (x_1, y_1) and (x_2, y_2) is $$(y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - x_2y_1) = 0.$$

1. Verify that this is the equation of a line. Where do you use the assumption that the two points are distinct?

   Let $a = y_1 - y_2$, $b = x_2 - x_1$ and $c = x_1y_2 - x_2y_1$. Then the equation becomes $ax + by + c = 0$. We are assuming that the points are distinct, so either $x_1 \neq x_2$ in which case $b \neq 0$, or $y_1 \neq y_2$ in which case $a \neq 0$, or both. So this is the equation of a line.

2. Verify that each of the two points satisfies the equation.

   Substitute $x = x_1$, $y = y_1$ into the equation to get $(y_1 - y_2)x_1 + (x_2 - x_1)y_1 + (x_1y_2 - x_2y_1)$. Multiply things out and simplify, and the result is 0. Substituting $x = x_2$, $y = y_2$ also results in 0, as you can check yourself.

3. Derive this formula by trying to solve the following two equations simultaneously for a, b and c, assuming that a and b are not both zero:

$$\begin{aligned} ax_1 + by_1 + c = 0 \\ ax_2 + by_2 + c = 0 \end{aligned}$$

Subtracting the second equation from the first yields $a(x_1 - x_2) + b(y_1 - y_2) = 0$. The points (x_1, y_1) and (x_2, y_2) are distinct, so either $x_1 \neq x_2$ or $y_1 \neq y_2$ or both. Assume first that $x_1 \neq x_2$. Then:

$$a = -b\frac{y_1 - y_2}{x_1 - x_2}.$$

Substituting into the first equation gives:

$$-b\frac{y_1 - y_2}{x_1 - x_2}x_1 + by_1 + c = 0.$$

So

$$c = b\frac{y_1 - y_2}{x_1 - x_2}x_1 - by_1.$$

So the equation of the line becomes:

$$-b\frac{y_1 - y_2}{x_1 - x_2}x + by + b\frac{y_1 - y_2}{x_1 - x_2}x_1 - by_1 = 0.$$

Multiply through by $x_1 - x_2$, to get:

$$-b(y_1 - y_2)x + b(x_1 - x_2)y + b(y_1 - y_2)x_1 - by_1(x_1 - x_2) = 0,$$

which simplifies to:

$$-b(y_1 - y_2)x + b(x_1 - x_2)y + b(y_1x_2 - x_1y_2) = 0.$$

This is the desired formula, multiplied by $-b$; i.e., a constant multiple of the formula.

Using similar calculations, assuming $y_1 \neq y_2$ also leads to a constant multiple of the desired formula.

Therefore, the only line determined by the two distinct points is the one specified by the formula (or a constant multiple thereof).

4. Explain how you can conclude from the previous problem that Axiom I-1 holds for $\mathbf{E}^2$.

   We know two things: (1) there is at least one line containing the two given points (e.g., given by the formula); and (2) any other equation of the form $ax + by + c = 0$ satisfied by the two given points must be a constant multiple of the given formula. Therefore there is one and only one line containing the two given points.

5. Suppose $A = (1, 2)$, $B = (1, 5)$, and $C = (2, -4)$. Use the formula to determine an equation for the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

3.1.2 Determinants

The following are formulas for *determinants* of arrays or *matrices* of numbers. We won't say more about determinants right now, but just learn the formulas:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (afh + bdi + ceg).$$

Two other equivalent formulas for 3×3 matrices are:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix}.$$

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{vmatrix} = a \begin{vmatrix} f & g & h \\ j & k & \ell \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & j & \ell \\ m & o & p \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & \ell \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}.$$

Another equivalent formula is:

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{vmatrix} = a \begin{vmatrix} f & g & h \\ j & k & \ell \\ n & o & p \end{vmatrix} - e \begin{vmatrix} b & c & d \\ j & k & \ell \\ n & o & p \end{vmatrix} + i \begin{vmatrix} b & c & d \\ f & g & h \\ n & o & p \end{vmatrix} - m \begin{vmatrix} b & c & d \\ f & g & h \\ j & k & \ell \end{vmatrix}.$$

1. Calculate the following determinants:

   (a)

$$\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$$

   (b)

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 4 & 3 \\ -2 & 0 & 5 \end{vmatrix}$$

   (c)

$$\begin{vmatrix} -1 & 1 & 2 & -3 \\ 0 & -2 & 4 & 5 \\ 3 & 0 & 0 & -4 \\ 2 & 6 & 10 & -7 \end{vmatrix}$$

3.1.3 Equations of Lines via Determinants

Determinants can be used to express concisely the equation of a line determined by two points:

> An equation of the line containing the distinct points (x_1, y_1) and (x_2, y_2) is
>
> $$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

1. Show that above statement is correct.

   Using the formula for the determinant, we get:

   $$xy_1 + x_1y_2 + x_2y - (xy_2 + x_1y + x_2y_1) = 0.$$

   Simplifying gives:

   $$(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1 = 0.$$

2. Suppose $A = (1, 2)$, $B = (1, 5)$ and $C = (2, -4)$. Use this version of the formula to determine an equation for the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

3.1.4 Testing Collinearity

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

1. Prove this statement. Suggestion: You might have to consider the special case that all three points are identical.

   If any two of the points are the same, you can do easy algebra to see that the determinant above equals 0. So assume that all three points are distinct. They are collinear if and only if the point (x_1, y_1) lies on the unique line determined by the other two points. This is true if and only if the point (x_1, y_1) satisfies the equation:

   $$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

   Substituting $x = x_1$, $y = y_1$ into the equation yields the desired result:

   $$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

2. Use this formula to show that the points $A = (1, 2)$, $B = (1, 5)$ and $C = (2, -4)$ are not collinear.

3. Use this formula to show that the points $A = (1, 2)$, $B = (23/2, 5)$ and $C = (2, -4)$ are collinear.

3.1.5 Intersections of Lines

From a previous theorem we know that if two lines intersect, then they intersect in exactly one point. Given two lines in $\mathbf{E}^2$, how can we compute the coordinates of that point? One method is to use *Cramer's Rule*:

> If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect, then their point of intersection is given by:
>
> $$x = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

1. Prove the above statement.

   Multiply $a_1x + b_1y + c_1 = 0$ by b_2 and $a_2x + b_2y + c_2 = 0$ by b_1 and subtract. The result is $(a_1b_2 - a_2b_1)x + b_2c_1 - b_1c_2 = 0$ or

   $$(a_1b_2 - a_2b_1)x = b_1c_2 - b_2c_1 = 0. \qquad (*)$$

   Now solve for x, which you can do if $a_1b_2 - a_2b_1 \neq 0$:

   $$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}.$$

   This matches the desired formula for x. The formula for y can be derived in a similar way.

   Note that if $a_1b_2 - a_2b_1 = 0$, then the left-hand side of $(*)$ is zero, indicating that (a_2, b_2) is a multiple of (a_1, b_1). If the right-hand side of $(*)$ is zero, then the two equations describe the same line and the lines are not different. If the right-hand side of $(*)$ is not zero, then the two equations have no solutions in common and the lines do not intersect.

2. What happens when you try to apply this formula to two lines that do not intersect?

   This was answered in the previous question.

3. Try to make sense of the following statement:

> If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect, then their point of intersection is given by:
>
> $$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Expanding this determinant gives:

$$a(b_1c_2 - c_1b_2) + b(a_2c_1 - a_1c_2) + c(a_1b_2 - a_2b_1) = 0. \qquad (**)$$

Divide through by $a_1b_2 - a_2b_1$ if this is nonzero. The coefficients of a and b then give the x and y coordinates, respectively, of the point of intersection. In this case, $(**)$ describes the set of all (a, b, c) such that the line $ax + by + c = 0$ passes through the point of intersection of the two given lines.

3.1.6 Parametric Equations of Lines

Here is another useful description of a line determined by two points.

> If (x_1, y_1) and (x_2, y_2) are two distinct points, then the line containing them is the set of points $\{(x_1, y_1) + t(u, v) : t \in \mathbf{R}\}$, where $u = x_2 - x_1$ and $v = y_2 - y_1$.

1. Prove that this description is correct; i.e., prove that this set is exactly the same as the set of points on the line containing the original two points, as given by the equation in Section 3.1.3.

   Define the two sets:

$$S = \{(x, y) : (y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1 = 0\},$$

$$S' = \{(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)) : t \in \mathbf{R}\}.$$

   We must show that $S = S'$.

   First take a point in S' and show that it satisfies the equation of S:

$$\begin{aligned} & (y_1 - y_2)(x_1 + t(x_2 - x_1)) + (x_2 - x_1)(y_1 + t(y_2 - y_1)) + x_1y_2 - x_2y_1 \\ = \ & x_1y_1 - x_1y_2 + t(y_1 - y_2)(x_2 - x_1) + x_2y_1 - x_1y_1 + t(x_2 - x_1)(y_2 - y_1) + x_1y_2 - x_2y_1 \\ = \ & 0. \end{aligned}$$

   Therefore $S' \subseteq S$.

   Now take a point (x, y) in S and show that it is in S'. We are looking for a value of t so that

$$\begin{aligned} x &= x_1 + t(x_2 - x_1) \qquad (*) \\ y &= y_1 + t(y_2 - y_1) \qquad (**) \end{aligned}$$

   Assume that $x_1 \neq x_2$. (The case that $y_1 \neq y_2$ is similar and we will omit the details.) Use $(*)$ to solve for t:

$$t = \frac{x - x_1}{x_2 - x_1}.$$

Let y' be the result of substituting this value of t into the right-hand side of $(**)$:

$$\begin{aligned} y' &= y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1) \\ &= \frac{x_2 y_1 - x_1 y_1 + (x - x_1)(y_2 - y_1)}{x_2 - x_1} \\ &= \frac{x_2 y_1 + x y_2 - x y_1 - x_1 y_2}{x_2 - x_1}. \end{aligned}$$

We must show that $y' = y$. Use $(y_1 - y_2)x + (x_2 - x_1)y + x_1 y_2 - x_2 y_1 = 0$ to solve for y:

$$y = \frac{x_2 y_1 - x_1 y_2 - x y_1 + x y_2}{x_2 - x_1}.$$

So $y = y'$ and the point (x, y) is seen to be in S' using the above value of t. Therefore $S \subseteq S'$.

2. What point on the line do you get when $t = 0$? When $t = 1$? When $t = 1/2$? When $t = 2$? When $t = -1$? Try plotting these points and explain their geometric relationship to the original two points.

   Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. The points you get from various values of t are:

   (a) $t = 0$: A.

   (b) $t = 1$: B:

   (c) $t = 1/2$: The midpoint of the line segment $\overline{AB}$.

   (d) $t = 2$: The point C so that B is the midpoint of the segment $\overline{AC}$.

   (e) $t = -1$: The point C so that A is the midpoint of the segment $\overline{BC}$.

3. Suppose $A = (1, 2)$, $B = (1, 5)$ and $C = (2, -4)$. Use this description to obtain the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

3.2 Some Trigonometry

3.2.1 Definitions of Sine and Cosine

The circle with center at the origin and radius of one unit is often called the *unit circle*. Recall that the equation of the unit circle is given by $x^2 + y^2 = 1$.

The trigonometric functions or ratios are often referred to as the *circular* functions. Let θ be an angle of rotation about the origin, measured from the positive x-axis, where a counterclockwise rotation produces a positive angle, as shown in Figure 1. The point $P(a, b)$ on the unit circle corresponds to θ.

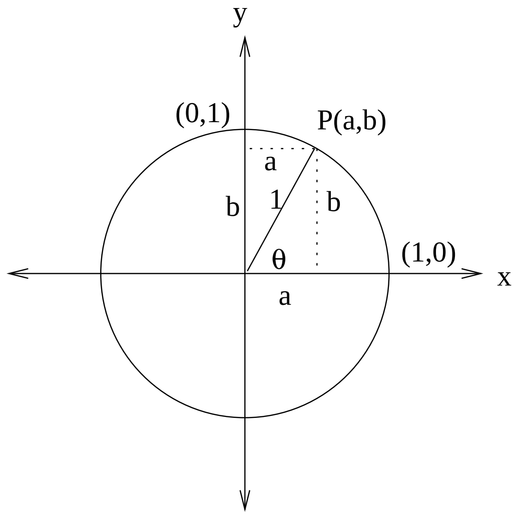


Figure 1: The Unit Circle — An Angle of Rotation

Definition: The *cosine* of θ, denoted $\cos\theta$, is the first, or x coordinate of the corresponding point P on the unit circle. In Figure 1, $a = \cos\theta$.

Definition: The *sine* of θ, denoted $\sin\theta$, is the second, or y coordinate of the corresponding point P on the unit circle. In Figure 1, $b = \sin\theta$.

1. Since $(\cos\theta, \sin\theta)$ is a point on the unit circle, what equation do these coordinates satisfy? (Note: this is a trig identity that you are familiar with!)

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

2. What is the largest value of $\sin\theta$? Give three values of θ where this maximum value is attained.

   The largest value is 1 and is attained, for example, at $\theta = 90°$, $450°$, and $-270°$.

3. What is the smallest value of $\sin\theta$? Give three values of θ where this minimum is attained.

   The smallest value is -1 and is attained, for example, at $\theta = 270°$, $630°$, and $-90°$.

4. Place your finger on the point $(1, 0)$, and trace the unit circle, counterclockwise, stopping at the point $(-1, 0)$. While doing this, how does the second coordinate of the point on the unit circle change? With the same question in mind, continue tracing around the circle.

   It begins at 0 when $\theta = 0°$, rises to 1 when $\theta = 90°$, drops to 0 when $\theta = 180°$, drops to -1 when $\theta = 270°$, and rises to 0 when $\theta = 360°$.

5. How does the first coordinate of the point change on this journey?

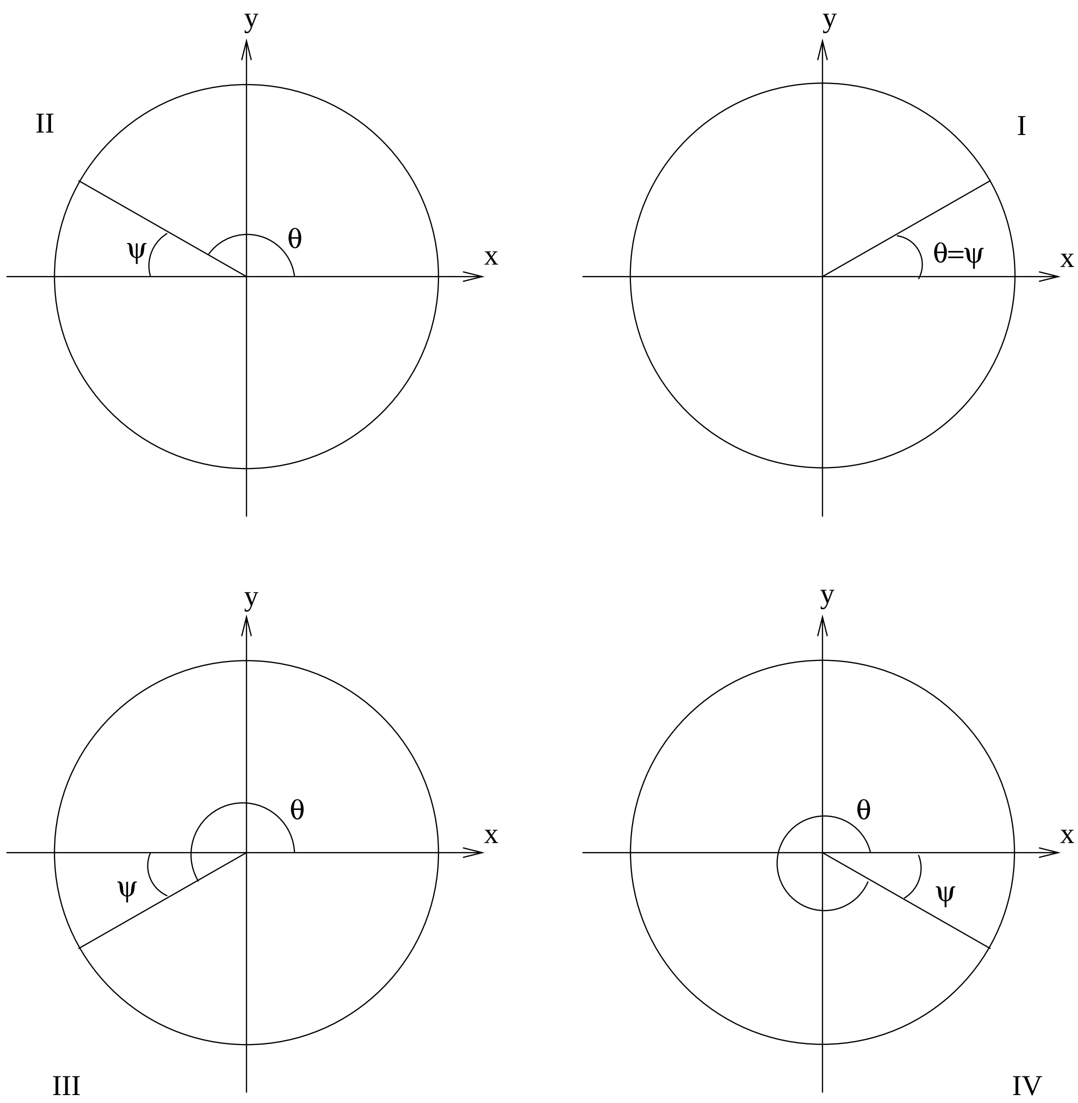
   It begins at 1 when $\theta = 0°$, drops to 0 when $\theta = 90°$, drops to -1 when $\theta = 180°$, rises to 0 when $\theta = 270°$, and rises to 1 when $\theta = 360°$.

6. Since the second coordinate of the point on the unit circle is the sine of the corresponding angle of rotation, and the first coordinate is the cosine of this angle, we can use the previous two problems to sketch the graphs of the sine and the cosine as a function of the angle. Sketch both graphs.

   See the Maple worksheet trig.ms.

7. Is circular a good description of the sine and cosine functions?

3.2.2 Angles in Different Quadrants



The above figure shows four possible positions of the angle θ—one in each of the four quadrants. Another angle, Ψ, is also marked. This is sometimes called the *reference angle* for θ. In each case, $\sin\theta = \pm\sin\Psi$ and $\cos\theta = \pm\cos\Psi$. Also, $\tan\theta = \pm\tan\Psi$, recalling that $\tan\theta = \sin\theta/\cos\theta$. The choice of $\pm$ depends upon the quadrant.

Fill in the chart:

1. Quadrant I
   (a) $\sin\theta = \sin\Psi$.
   (b) $\cos\theta = \cos\Psi$.
   (c) $\tan\theta = \tan\Psi$.
2. Quadrant II
   (a) $\sin\theta = \sin\Psi$.
   (b) $\cos\theta = -\cos\Psi$.
   (c) $\tan\theta = -\tan\Psi$.
3. Quadrant III
   (a) $\sin\theta = -\sin\Psi$.
   (b) $\cos\theta = \tan\Psi$.
   (c) $\tan\theta = -\cos\Psi$.
4. Quadrant IV
   (a) $\sin\theta = -\sin\Psi$.
   (b) $\cos\theta = \cos\Psi$.
   (c) $\tan\theta = -\tan\Psi$.

One mnemonic device to remember the signs of the trig functions sine, cosine and tangent (sine/cosine) in the four quadrants is **A S**mart **T**rig **C**lass:

A **All** three trig functions are positive in Quadrant I.

S Only **Sine** is positive in Quadrant II; the others are negative.

T Only **Tangent** is positive in Quadrant III; the others are negative.

C Only **Cosine** is positive in Quadrant IV; the others are negative.

3.2.3 Some Familiar Angles

Here is a mnemonic to remember the sine and cosine of some common angles:

θ	0°	30°	45°	60°	90°
$\sin\theta$	$\sqrt{0}/2$	$\sqrt{1}/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2$
$\cos\theta$	$\sqrt{4}/2$	$\sqrt{3}/2$	$\sqrt{2}/2$	$\sqrt{1}/2$	$\sqrt{0}/2$

1. Convince yourself that these values are correct by considering $45^\circ - 45^\circ - 90^\circ$ triangles and $30^\circ - 60^\circ - 90^\circ$ triangles.

3.2.4 Some Basic Identities

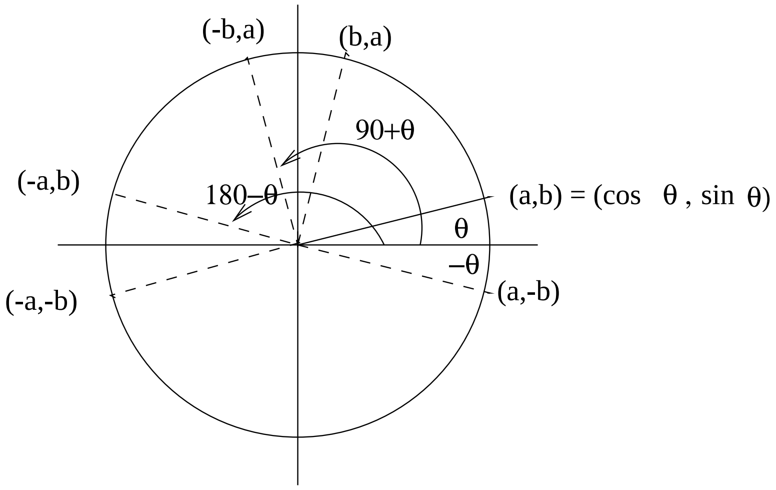


Figure 2: Unit Circle — $a = \cos\theta$, $b = \sin\theta$

1. Using the coordinates a and b, fill in the missing coordinates in Figure 2. Since $a = \cos\theta$ and $b = \sin\theta$, we easily get the sine and cosine of other angles, such as $-\theta$, $90° + \theta$, $180° + \theta$, $90° - \theta$, and $180° - \theta$. For example, $\sin(-\theta) = -b = -\sin\theta$. Using Figure 2, complete and/or verify the following identities.

   (a) $\cos(-\theta) = a = \cos\theta$.

   (b) $\sin(-\theta) = -b = -\sin\theta$.

   (c) $\cos(90° + \theta) = -b = -\sin\theta$.

   (d) $\sin(90° + \theta) = a = \cos\theta$.

   (e) $\cos(180° + \theta) = -a = -\cos\theta$.

   (f) $\sin(180° + \theta) = -b = -\sin\theta$.

   (g) $\cos(90° - \theta) = b = \sin\theta$.

   (h) $\sin(90° - \theta) = a = \cos\theta$.

   (i) $\cos(180° - \theta) = -a = -\cos\theta$.

   (j) $\sin(180° - \theta) = b = \sin\theta$.

3.2.5 Polar Coordinates of Points in $\mathbf{E}^2$

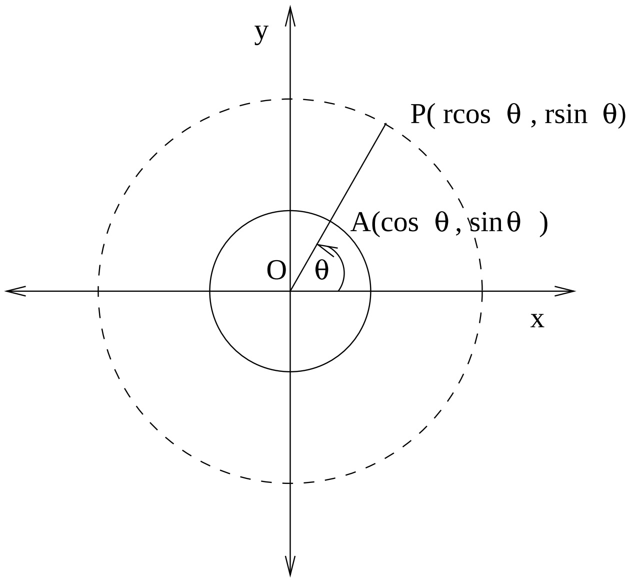
We assign coordinates to points in $\mathbf{E}^2$ so that we can locate them. Let P be a point in $\mathbf{E}^2$, and $\overline{OP}$ be the line segment between P and the origin. Let r be the distance between P and the origin. (So r is the length of $\overline{OP}$.) Let θ be the angle, measured in the counterclockwise direction, that $\overline{OP}$ makes with the x-axis. (See the figure below.)

The point P lies on the circle of radius r, centered at the origin, given by the equation $x^2 + y^2 = r^2$.

Verify that $x = r\cos\theta$ and $y = r\sin\theta$ satisfy this equation.

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(cos^2\theta + \sin^2\theta) = r^2.$$

(r, θ) are called the *polar coordinates* for the point P.



1. Explain exactly when two sets of polar coordinates (r, θ) and (r', θ') describe precisely the same point.

(a) $r = r' = 0$, and θ and θ' are arbitrary.

(b) $r \neq 0$, $r = r'$, and $\theta' = \theta + k360°$, where k is an integer.

(c) $r \neq 0$, $r' = -r$, and $\theta' = \theta + 180° + k360°$, where k is an integer.

3.2.6 Alternate Definitions of Sine and Cosine

If θ is an acute angle, then we can draw a right triangle, as in Figure 3 (A). The side opposite the angle θ is labeled b, the side adjacent to θ is labeled a and the hypotenuse is labeled c. In Figure 3 (B), we have placed the vertex A at the origin, and the adjacent side on the x-axis. The vertex B is at the point (a, b).

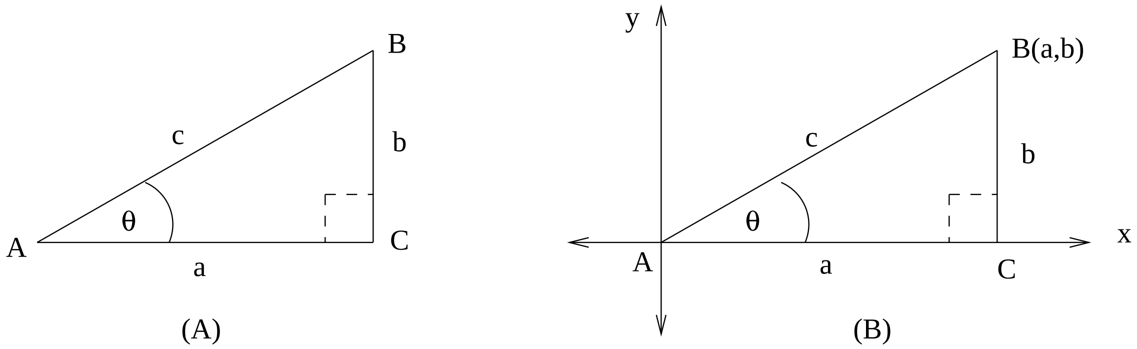


Figure 3: $\theta < 90°$

Alternate definitions for $\sin\theta$ and $\cos\theta$ are:

$$\sin\theta = \frac{\textit{opposite side}}{\textit{hypotenuse}} = \frac{opp}{hyp} = \frac{\mathbf{O}}{\mathbf{H}} = \frac{b}{c}$$

$$\cos\theta = \frac{\textit{adjacent side}}{\textit{hypotenuse}} = \frac{adj}{hyp} = \frac{\mathbf{O}}{\mathbf{H}} = \frac{a}{c}$$

Note that when $c = 1$, the point $B(a, b)$ lies on the unit circle, and $\sin\theta = \frac{b}{c} = \frac{b}{1} = b$ is the second coordinate of the point $B(a, b)$, while $\cos\theta = \frac{a}{c} = \frac{a}{1} = a$ is the first coordinate of this point. Thus these alternate definitions are consistent with the old definitions of sine and cosine.

3.2.7 Definitions of Tangent, Secant, and Cosecant

Let θ denote an angle.

Definition: The *tangent* of θ, denoted $\tan\theta$ is

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}.}$$

Definition: The *secant* of θ, denoted $\sec\theta$ is

$$\boxed{\sec\theta = \frac{1}{\cos\theta}.}$$

Definition: The *cosecant* of θ, denoted $\csc\theta$ is

$$\boxed{\csc\theta = \frac{1}{\sin\theta}.}$$

1. We left out the *cotangent* of θ. Give an appropriate definition as above.

$$\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}.}$$

2. Describe the domain and the range of each of the six trig functions.

   (a) $\sin\theta$: Domain equals $\mathbf{R}$. Range equals $[-1, 1]$.
   (b) $\cos\theta$: Domain equals $\mathbf{R}$. Range equals $[-1, 1]$.
   (c) $\tan\theta$: Domain equals $\mathbf{R} \setminus \{90° + n180° : n \in \mathbf{Z}\}$. Range equals $\mathbf{R}$.
   (d) $\cot\theta$: Domain equals $\mathbf{R} \setminus \{n180° : n \in \mathbf{Z}\}$. Range equals $\mathbf{R}$.
   (e) $\sec\theta$: Domain equals $\mathbf{R} \setminus \{90° + n180° : n \in \mathbf{Z}\}$. Range equals $\mathbf{R} \setminus (-1, 1)$.
   (f) $\csc\theta$: Domain equals $\mathbf{R} \setminus \{n180° : n \in \mathbf{Z}\}$. Range equals $\mathbf{R} \setminus (-1, 1)$.

   See also the Maple worksheet trig.ms.

3. Using the alternate definition of sine and cosine as ratios of sides of right triangles, give alternate definitions of tangent, secant, cosecant, and cotangent.

   (a) $\tan\theta = \frac{\mathbf{O}}{\mathbf{A}}$.
   (b) $\cot\theta = \frac{\mathbf{A}}{\mathbf{O}}$.
   (c) $\sec\theta = \frac{\mathbf{H}}{\mathbf{A}}$.
   (d) $\csc\theta = \frac{\mathbf{H}}{\mathbf{O}}$.

4. **O**scar **H**ad **A** **H**eap **O**f **A**pples is a mnemonic to remember that $\sin = \frac{O}{H}$, $\cos = \frac{A}{H}$, and $\tan = \frac{O}{A}$. Make up your own mnemonic.

5. Starting with the known trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ we can derive two more identities by dividing both sides of this equation by $\cos^2\theta$ and by $\sin^2\theta$. Try this to find one identity involving $\tan^2\theta$ and another involving $\cot^2\theta$.

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

3.2.8 The Area of a Triangle Using Sines

It's time to put some algebra and trigonometry to use. In this problem we will use the triangle in Figure 4. In this triangle all angles have measure less than 90°, however, the results hold true for general triangles.

The lengths of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ are a, b and c, respectively. Segment $\overline{AD}$ has length c' and $\overline{DB}$ length c''. Segment $\overline{CD}$ is the altitude of the triangle from C, and has length h.

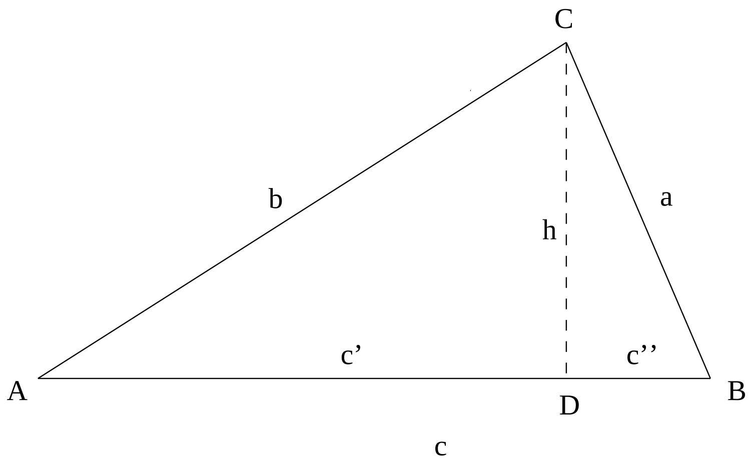


Figure 4: Triangle ABC

The usual formula for the area of a triangle is $\frac{1}{2}$(base)(height).

1. Using the given labeling, Area$(ABC) = \frac{1}{2}ch$.
2. Since triangle ADC is a right triangle, $\sin A = h/b$ so $h = b \sin A$.
3. Thus,

$$\boxed{\text{Area}(ABC) = \frac{1}{2}bc\sin A}$$

4. What is a formula for Area(ABC) using $\sin B$? Using $\sin C$?? (Note: you will have to use the altitude from A or B).

$$\boxed{\text{Area}(ABC) = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C}$$

The conclusion we have made is that the area of a triangle is one-half the product of the lengths of any two sides and the sine of the included angle.

3.2.9 The Law of Sines

Using the triangle in Section 3.2.8, the *Law of Sines* is:

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

The result holds for arbitrary triangles, but we shall prove it for the triangle ABC in Section 3.2.8.

1. We showed that the area of this triangle was given by three different formulas. What are they?

$$\text{Area}(ABC) = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C.$$

2. From these three formulas, prove the Law of Sines.

   Divide the above equation through by $\frac{1}{2}abc$.

3.2.10 The Law of Cosines

Using the triangle in Section 3.2.8, the *Law of Cosines* is:

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

1. Show that $c' = b\cos A$.

   Observe that $\cos A = c'/b$ from triangle ACD.

2. Verify that $c'' = c - c'$.

   This is clear from the diagram.

3. Verify that $h^2 = b^2 - (c')^2$.

   This follows from the Pythagorean Theorem applied to triangle ACD.

4. Apply the Pythagorean Theorem to triangle CDB, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.

$$\begin{aligned} a^2 &= h^2 + (c'')^2 \\ &= b^2 - (c')^2 + (c - c')^2 \\ &= b^2 - (c')^2 + c^2 - 2cc' + (c')^2 \\ &= b^2 + c^2 - 2cc' \\ &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

5. One last question: What happens when you Apply the Law of Cosines in the case that $\angle A$ is a right angle?

   In this case $\cos A = 0$ and you get $a^2 = b^2 + c^2$, which is the Pythagorean Theorem.

3.2.11 Addition Formulas

There are very useful formulas for the cos and the sin of the sum of two angles α and β:

$$\boxed{\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}}$$

Here we will derive these formulas for the case where $A + B < 180°$. In the figures below, $\overline{PA}$ is perpendicular to the x-axis, and $\overline{PB}$ is perpendicular to $\overline{OB}$. Thus triangle OBP is a right triangle in both pictures.

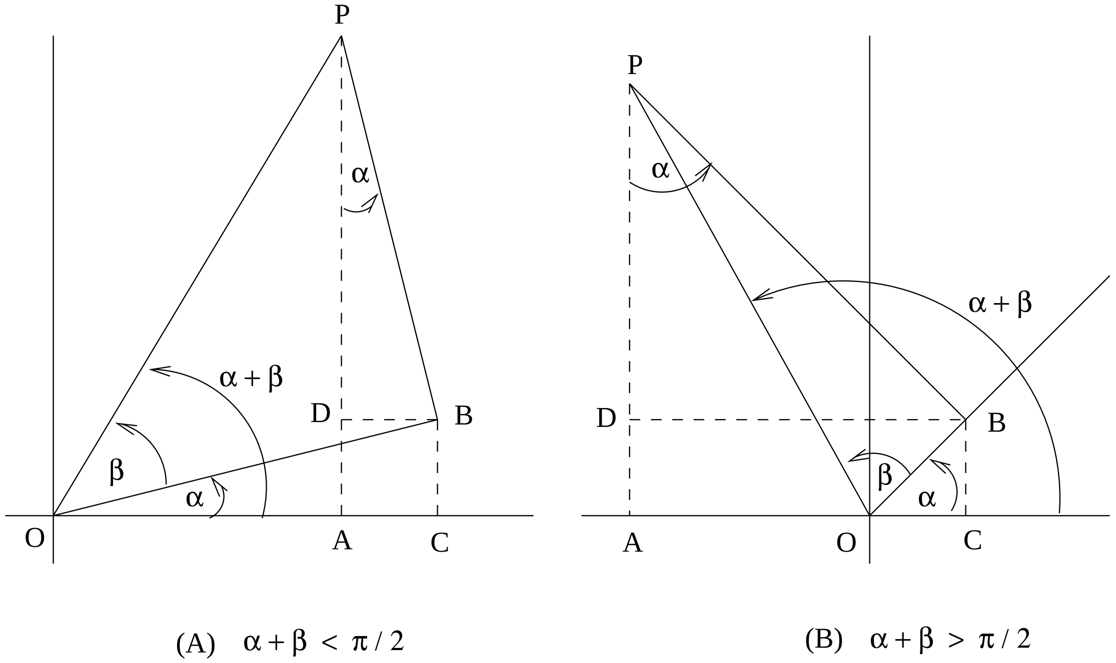


Figure 5: $\alpha + \beta$

First we gather some facts from Figure 5:

1. $AP = AD + DP$.
2. $AD = CB$. Thus $AD + DP = CB + DP$.
3. $OA = OC - AC$.
4. $DB = AC$. Thus $OC - AC = OC - DB$.
5. $\sin\alpha = \frac{DB}{PB}$ (using triangle PDB).
6. $\cos\alpha = \frac{DP}{PB}$ (using triangle PDB).
7. Using triangle OCB, $\sin\alpha = \frac{CB}{OB}$.
8. Using triangle OCB, $\cos\alpha = \frac{OC}{OB}$.
9. $\sin\beta = \frac{PB}{OP}$ and $\cos\beta = \frac{OB}{OP}$ (using triangle OBP in (A).
10. Using facts 5,8, and 9,

    $$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{OC}{OB}\frac{OB}{OP} - \frac{DB}{PB}\frac{PB}{OP} = \frac{OC-DB}{OP}. \ (*)$$

    Using Figure 5 (A),

    $\cos(\alpha+\beta) = \frac{OA}{OP} = \frac{OC-AC}{OP}$ using fact 3 above

    $= \frac{OC-DB}{OP}$ using fact 4

    $= \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (see $(*)$)

    which proves the first addition formula!
11. Using facts 6,7, and 9,

    $$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{CB}{OB}\frac{OB}{OP} + \frac{DP}{PB}\frac{PB}{OP} = \frac{CB+DP}{OP}. \ (**)$$

    Using Figure 5 (A),

    $\sin(\alpha+\beta) = \frac{AP}{OP} = \frac{AD+DP}{OP}$ using fact 1 above

    $= \frac{CB+DP}{OP}$ using fact 2

    $=\sin\alpha\cos\beta + \cos\alpha\sin\beta$ (see $(**)$)

    which proves the second addition formula!

3.2.12 Heron's Formula

In this section, we will put some of our results together to prove yet another formula for the area of a triangle.

For notation, refer to Figure 4 in Section 3.2.8.

Let $s = \frac{1}{2}(a+b+c)$ (s is half the perimeter). Heron's Formula for the area of triangle ABC is:

$$\boxed{\text{area}(ABC) = \sqrt{s(s-a)(s-b)(s-c)}}$$

First we gather some facts:

1. $\text{area}(ABC) = \frac{1}{2}bc\sin A$ (See Section 3.2.8).
2. $\sin^2 A + \cos^2 A = 1$, thus $\sin A = \sqrt{1-\cos^2 A}$.
3. $a^2 = b^2 + c^2 - 2bc\cos A$, thus
$$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$
(This is the Law of Cosines, proven in Section 3.2.10.)
4. $s - a = \frac{1}{2}(-a+b+c)$, $s-b = \frac{1}{2}(a-b+c)$, and $s-c = \frac{1}{2}(a+b-c)$.

Thus, area$(ABC) = \frac{1}{2}bc\sin A$ (fact 1)

$$= \tfrac{1}{2}\sqrt{1-\cos^2 A} \text{ (fact 2)}$$

$$= \tfrac{1}{2}bc\sqrt{1-\left(\tfrac{b^2+c^2-a^2}{2bc}\right)^2} \text{ (fact 3)}$$

$$= \tfrac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}$$

$$= \tfrac{1}{4}\sqrt{4b^2c^2-(b^4+c^4+a^4+2b^2c^2-2a^2b^2-2a^2c^2)}$$

$$= \tfrac{1}{4}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4} \qquad (*)$$

Now we work on the right hand side of Heron's Formula:

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\tfrac{1}{2}(a+b+c)\tfrac{1}{2}(-a+b+c)\tfrac{1}{2}(a-b+c)\tfrac{1}{2}(a+b-c)}$$

$$= \tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \qquad (**)$$

To finish the proof, we must show that $(*) = (**)$.

1. Show that $(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$. Just multiply out the left-hand side carefully.

3.2.13 A Cosine Formula

In this section we remind you of a nice formula to get the cosine of the angle using coordinates of points.

Assume that you have triangle ABC such that the coordinates of the three (distinct) points A, B, and C are $(0,0)$, (x_1, y_1), and (x_2, y_2), respectively. The Law of Cosines can be used to prove that

$$\boxed{\cos A = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.}$$

1. Use the Law of Cosines to prove this formula. Recall that the length of a line segment joining points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$$\begin{aligned}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - [(x_2 - x_1)^2 + (y_2 - y_1)^2]}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\
&= \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2 - (x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2)}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\
&= \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.
\end{aligned}$$

3.2.14 Determinantal Area Formula

Referring to the triangle ABC of the previous section, we can prove another area formula.

$$\text{area}(ABC) = \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}\left\| \begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array} \right\|$$

(I am using the notation $\|\cdot\|$ to denote the absolute value of the determinant.)

1. Use $\text{area}(ABC) = \frac{1}{2}bc\sin A$, the cosine formula from the previous section, and $\sin^2 A + \cos^2 A = 1$ to prove this formula.

$$\begin{aligned}
\text{area}(ABC) &= \tfrac{1}{2}bc\sin A \\
&= \tfrac{1}{2}bc\sqrt{1-\cos^2 A} \\
&= \frac{1}{2}\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\sqrt{1-\left(\frac{x_1x_2+y_1y_2}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}}\right)^2} \\
&= \frac{1}{2}\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\sqrt{\frac{(x_1^2+y_1^2)(x_2^2+y_2^2)-(x_1x_2+y_1y_2)^2}{(x_1^2+y_1^2)(x_2^2+y_2^2)}} \\
&= \tfrac{1}{2}\sqrt{x_1^2x_2^2+x_1^2y_2^2+x_2^2y_1^2+y_1^2y_2^2-(x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2)} \\
&= \tfrac{1}{2}\sqrt{x_1^2y_2^2-2x_1x_2y_1y_2+x_2^2y_1^2} \\
&= \tfrac{1}{2}\sqrt{(x_1y_2-x_2y_1)^2} \\
&= \tfrac{1}{2}|x_1y_2-x_2y_1|.
\end{aligned}$$

3.3 Analytical Euclidean Space $\mathbf{E}^3$

Points in $\mathbf{E}^3$ are ordered triples (x, y, z) of real numbers (called *Cartesian coordinates*). Lines are sets of the form $\{(x_1, y_1, z_1) + t(u, v, w) : t \in \mathbf{R}\}$, where $(u, v, w) \neq (0, 0, 0)$. Planes are sets of the form $\{(x, y, z) : ax + by + cz + d = 0\}$, where $(a, b, c) \neq (0, 0, 0)$.

1. Cylindrical coordinates.

   (a) Explain how a point (x, y, z) in $\mathbf{E}^3$ can be described by cylindrical coordinates (r, θ, z).
   To convert from cylindrical coordinates to Cartesian coordinates,

   $$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

   (b) Explain exactly when two sets of cylindrical coordinates (r, θ, z) and (r', θ', z') describe precisely the same point.
   For (r, θ, z) and (r', θ', z') to give the same (x, y, z):

   i. z' must equal z.
   ii. If $r = 0$ then $r' = 0$, and θ and θ' can be arbitrary.
   iii. If $r' \neq 0$ then either
      A. $r' = r$ and $\theta' = \theta + n \cdot 360°$ where n is an integer; or
      B. $r' = -r$ and $\theta' = \theta + 180° + n \cdot 360°$ where n is an integer.

2. Spherical coordinates.

   (a) Explain how a point (x, y, z) in $\mathbf{E}^3$ can be described by spherical coordinates $(\rho, \theta\, \phi)$.
   To convert from spherical coordinates to Cartesian coordinates,

   $$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

(b) Explain exactly when two sets of spherical coordinates (ρ, θ, ϕ) and (ρ', θ', ϕ') describe precisely the same point.

For (ρ, θ, ϕ) and $(\rho', \theta' \phi')$ to give the same (x, y, z):

i. If $\rho = 0$ then $\rho' = 0$ and $\theta, \theta', \phi, \phi'$ are arbitrary.

ii. If $\rho \neq 0$ and $\sin \phi = 0$ ($\phi = m \cdot 180°$ where m is an integer) then either

A. $\rho' = \rho$, $\phi' = \phi + n \cdot 360°$ where n is an integer, and θ, θ' are arbitrary; or

B. $\rho' = -\rho$, $\phi' = \phi + 180° + n \cdot 360°$ where n is an integer, and θ, θ' are arbitrary.

iii. If $\rho \neq 0$ and $\sin \phi \neq 0$ then

A. $\rho' = \rho$, $\phi' = \phi + m \cdot 360°$, $\theta' = \theta + n \cdot 360°$, where m and n are integers; or

B. $\rho' = \rho$, $\phi' = -\phi + m \cdot 360°$, $\theta' = \theta + 180° + n \cdot 360°$, where m and n are integers; or

C. $\rho' = -\rho$, $\phi' = \phi + 180° + m \cdot 360°$, $\theta' = \theta + n \cdot 360°$, where m and n are integers; or

D. $\rho' = -\rho$, $\phi' = -\phi + 180° + m \cdot 360°$, $\theta' = \theta + 180° + n \cdot 360°$, where m and n are integers.

3. Locations on the earth are specified by degrees latitude and longitude, where latitude ranges from 0° to 90° S and from 0° to 90° N, and longitude ranges from 0° to 180° E and from 0° to 180° W. The great semicircle specified by 0° longitude is known as the *Prime Meridian.*

(a) Assuming that the North Pole has Cartesian coordinates $(0, 0, 1)$, the South Pole has Cartesian coordinates $(0, 0, -1)$, and the Prime Meridian crosses the equator at the point with Cartesian coordinates $(1, 0, 0)$, explain how to convert from spherical coordinates to latitude and longitude.

If $0 \leq \theta \leq 180°$ then longitude is θ East. If $180° \leq \theta \leq 360°$ then longitude is $360° - \theta$ West. If $0 \leq \phi \leq 90°$ then latitude is $90° - \phi$ North. If $90° \leq \phi \leq 180°$ then latitude is $\phi - 90°$ South.

(b) Explain how to convert from latitude and longitude to spherical coordinates.

For α East longitude, set $\theta = \alpha$. For α West longitude, set $\theta = 360° - \alpha$. For β North latitude, set $\phi = 90° - \beta$. For β South latitude, set $\phi = \beta + 90°$.

4. Determine nice coordinates for the eight vertices (corners) of a cube such that the origin $(0, 0, 0)$ is at the center of the cube.

One answer is the eight points with coordinates $(\pm 1, \pm 1, \pm 1)$.

5. Determine nice coordinates for the six vertices of an *octahedron*: a solid whose faces consist of eight equilateral triangles, with exactly four triangles meeting at each vertex. Assume that the origin is at the center of the octahedron.

   One answer is the six points with coordinates $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.

6. Explain why the following points form the three corners of an equilateral triangle: $(1, 0, 0)$, $(\cos 120°, \sin 120°, 0)$, $(\cos 240°, \sin 240°, 0)$.

   Converting to polar coordinates, one sees that these are three equally spaced points on the unit circle centered at the origin.

7. Using a construction kit like Polydron, Roger's Connection, Zometool, or Googolplex, try to build as many objects as you can with the following properties:

   (a) Each face (side) is an equilateral triangle.

   (b) Each equilateral triangle is joined to an equilateral triangle along each of its three faces (these three adjoining triangle are its *neighbors*).

   (c) No two neighbors are coplanar.

   (d) The plane determined by any of the faces does not intersect the rest of the object anywhere else (this forces the object to be *convex*).

   How many such objects can you construct? These polyhedra are called the *deltahedra*. Determine exact coordinates (e.g., perhaps involving sin, cos, $\sqrt{\ }$, etc.—not just numerical approximations) for the corners of each object.

8. Prove algebraically that each incidence axiom I-1 through I-5 holds in $\mathbf{E}^3$. Watch out for various special cases in the event that you divide by an expression that might equal 0.

9. Assume that (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are three distinct noncollinear points. Prove:

   An equation of the plane containing (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

3.4 Other Geometrical Worlds

3.4.1 The Sphere $\mathbf{S}^2$

POINTS in $\mathbf{S}^2$ are ordered triples (x, y, z) such that $x^2+y^2+z^2 = 1$. A LINE is a set of points in $\mathbf{S}^2$ that also satisfy an equation of the form $ax + by + cz = 0$, where $(a, b, c) \neq (0, 0, 0)$; i.e., intersections of $\mathbf{S}^2$ with planes in $\mathbf{R}^3$ through the origin.

POINTS in $\mathbf{S}^2$ can also be described by spherical coordinates $(1, \theta, \phi)$; i.e., points with spherical coordinates in which $\rho = 1$.

3.4.2 The Punctured Sphere $\mathbf{U}^2$

POINTS in $\mathbf{U}^2$ are ordered triples (x, y, z) such that $x^2 + y^2 + z^2 = 1$, except for the triple $N = (0, 0, 1)$ which is excluded. A LINE is a set of points in $\mathbf{U}^2$ that also lie on a plane in $\mathbf{R}^3$ that passes through N and is not tangent to the sphere. What is the general equation of such a plane?

3.4.3 The Open Hemisphere $\mathbf{H}^2$

POINTS in $\mathbf{H}^2$ are ordered triples (x, y, z) such that $x^2 + y^2 + z^2 = 1$ and $z > 1$. A LINE is a set of points in $\mathbf{H}^2$ that also lie on a plane in $\mathbf{R}^3$ that is perpendicular to the xy-plane and intersects $\mathbf{H}^2$. What is the general equation of such a plane?

3.4.4 The Projective Plane $\mathbf{P}^2$

POINTS in $\mathbf{P}^2$ are ordinary lines in $\mathbf{R}^3$ that pass through the origin. So each POINT is a set of the form $\{t(x, y, z) : t \in \mathbf{R}\}$, for some ordered triple $(x, y, z) \neq (0, 0, 0)$. LINES are ordinary planes in $\mathbf{R}^3$ that pass through the origin. So each LINE is a set of the form $\{(x, y, z) : ax + by + cz = 0\}$, where $(a, b, c) \neq (0, 0, 0)$.

3.4.5 Analytical Euclidean 4-Space: $\mathbf{E}^4$

Points in $\mathbf{E}^4$ are ordered quadruples (x_1, x_2, x_3, x_4) of real numbers. Lines are sets of the form $\{(x_1, x_2, x_3, x_4) + t(u_1, u_2, u_3, u_4) : t \in \mathbf{R}\}$, where $(u_1, u_2, u_3, u_4) \neq (0, 0, 0, 0)$. Planes are sets of the form $\{(x_1, x_2, x_3, x_4) + s(u_1, u_2, u_3, u_4) + t(v_1, v_2, v_3, v_4) : s, t \in \mathbf{R}\}$, where $(u_1, u_2, u_3, u_4) \neq (0, 0, 0, 0)$, $(v_1, v_2, v_3, v_4) \neq (0, 0, 0, 0)$, and (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) are not multiples of each other.

3.4.6 Analytical Euclidean n-Space: $\mathbf{E}^n$

Assume n is an integer greater than 2. Points in $\mathbf{E}^n$ are ordered n-tuples $x = (x_1, x_2, x_3, \ldots, x_n)$. Lines are sets of the form $\{x + tu : t \in \mathbf{R}\}$, where x is an n-tuple and u is a nonzero n-tuple. Planes are sets of the form $\{x + su + tv : s, t \in \mathbf{R}\}$, where x is an n-tuple, u and v are nonzero n-tuples, and u and v are not multiples of each other.