

Some Additional Notes on Polyhedra

Platonic Solids

A *Platonic* or *regular* solid is a polyhedron in which all faces are congruent regular polygons, and the same number of polygons meet at each vertex. Like the book, I will first focus on convex polyhedra.

Here is the proof that I gave in class that there are only five Platonic solids: Suppose that there are V vertices, each of degree r , and F faces, each a k -gon. Summing the degrees of the vertices counts every edge twice, so $rV = 2E$ or

$$V = \frac{2E}{r}.$$

Summing the numbers of edges in each polygon also counts every edge twice, so $kF = 2E$ or

$$F = \frac{2E}{k}.$$

Now substitute into Euler's Formula:

$$\begin{aligned} 2 &= V - E + F \\ &= \frac{2E}{r} - E + \frac{2E}{k} \\ &= E \left(\frac{2}{r} - 1 + \frac{2}{k} \right) \\ &= E \left(\frac{2k - rk + 2r}{rk} \right). \end{aligned}$$

Solving for E :

$$\begin{aligned} E &= \frac{2rk}{2k + 2r - rk} \\ &= \frac{2rk}{4 - (k-2)(r-2)}. \end{aligned}$$

Since the result must be positive, we conclude that $(k-2)(r-2)$ must be less than 4. Of course, k and r must each be at least 3. So the choices for $(k-2)$ and $(r-2)$ are:

$k-2$	$r-2$
1	1
1	2
1	3
2	1
3	1

This gives the following possible values of k and r , listed below with the name of the Platonic solid:

k	r	Platonic solid
3	3	tetrahedron
3	4	octahedron
3	5	icosahedron
4	3	cube
5	2	dodecahedron

It remains to show that these five polyhedra actually exist. We can read Book XIII of *Euclid's Elements*—see, for example,

<http://aleph0.clarku.edu/~djoyce/java/elements/elements.html>.

Or we can work directly by finding coordinates for the vertices. The cube and octahedron are easiest. A tetrahedron can be constructed within a cube by selecting every other vertex. An icosahedron can be constructed by using three self-intersecting golden rectangles to determine the vertices—see, for example, <http://en.wikipedia.org/wiki/Icosahedron>. A dodecahedron can then be constructed by placing a vertex in the center of each triangle of an icosahedron.

The notion of regular polyhedron has been extended to the nonconvex case, resulting in four more *Kepler-Poinsot* solids. The definition has also been extended to higher dimensions. In four dimensions there are six regular convex polyhedra, and in every dimension thereafter there are only three.

Semiregular and Archimedean Solids.

A *semiregular* solid is a polyhedron composed of two or more regular polygons with the same sequence of polygons meeting at each vertex. Further, there is a requirement that for every pair of vertices there is a symmetry of the polyhedra that maps one of these vertices onto the other.

We can use Euler's Formula to figure out the numbers of vertices, edges, and polygons from the combination of faces meeting at each vertex. Let $S = (p_1, p_2, \dots, p_q)$ be the given sequence, and m_p be the number of times the number p appear in the sequence. Every vertex is incident with m_p p -gons, and every p -gon has p vertices. Then $m_p V$ counts every p -gon p times, and the total number of p -gons is given by

$$F_p = m_p V / p.$$

Every vertex is incident with q edges, and every edge is incident with 2 vertices. So qV counts every edge twice, and

$$E = qV / 2.$$

Now

$$\begin{aligned} 2 &= V - E + F \\ &= V - qV/2 + \sum_p F_p \\ &= V - qV/2 + \sum_p m_p V/p. \end{aligned}$$

where the sums are taken over all distinct numbers p in S . Thus

$$2 = V \left(1 - q/2 + \sum_p m_p/p \right),$$

and we can now solve for V . Once we have V , we can use the earlier equations to solve for E and F_p . The total number of faces will be the sum of the F_p .

It turns out that there are two infinite families of semiregular solids—the *prisms* with sequence $(4, 4, p)$, $p \geq 3$, $p \neq 4$; and the *antiprisms* with sequence $(3, 3, 3, p)$, $p \geq 4$. Apart from these there are only 13 others, known as the *Archimedean solids*. One well-known example is the soccer ball, otherwise known as the truncated icosahedron, with sequence $(5, 6, 6)$. This is also the structure of Buckminsterfullerene, a carbon molecule C_{60} composed of 60 carbon atoms.