CHAPTER EIGHT

The Game of Hex

IT IS something of an occasion these days when someone invents a mathematical game that is both new and interesting. Such a game is Hex, introduced 15 years ago at Niels Bohr's Institute for Theoretical Physics in Copenhagen. It may well become one of the most widely played and thoughtfully analyzed new mathematical games of the century.

Hex is played on a diamond-shaped board made up of hexagons [*see Fig. 33*]. The number of hexagons may vary, but the board usually has 11 on each edge. Two opposite sides of the diamond are labeled "black"; the other two sides are "white." The hexagons at the corners of the diamond belong to either side. One player has a supply of black pieces; the other, a supply of white pieces. The players alternately place one of their pieces on any one of the hexagons, provided the cell is not already occupied by another piece. The objective

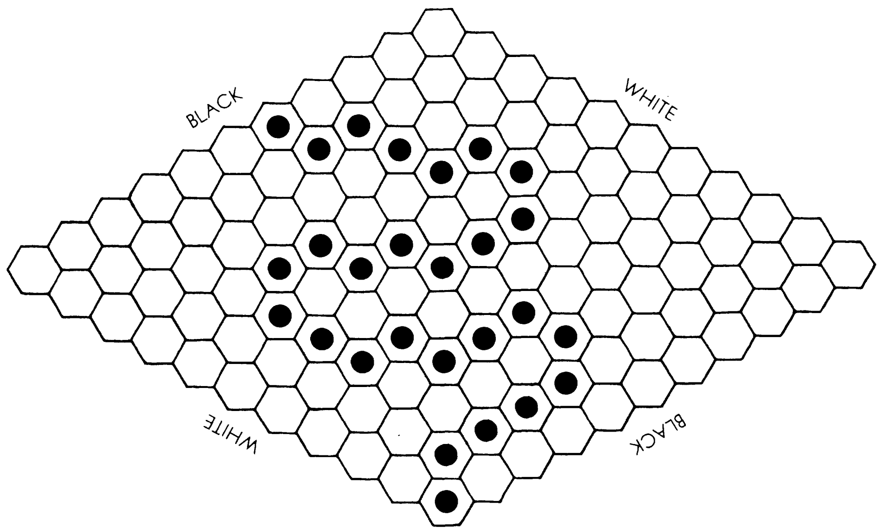

FIG. 33.
A winning chain for "black" on a Hex board with 11 hexagons on each side.

of "black" is to complete an unbroken chain of black pieces between the two sides labeled "black." "White" tries to complete a similar chain of white pieces between the sides labeled "white."

The chain may freely twist and turn; an example of a winning chain is shown in Figure 33. The players continue placing their pieces until one of them has made a complete chain. The game cannot end in a draw, because one player can block the other only by completing his own chain. These rules are simple, yet Hex is a game of surprising mathematical subtlety.

Hex was invented by Piet Hein, who must surely be one of the most remarkable men in Denmark. Hein began his career as a student at the Institute for Theoretical Physics; then his industrial inventions switched him to engineering, where he remained until the Germans invaded Denmark in

1940. Because Hein was the head of an anti-Nazi group, he was forced to go underground. After the war he became well known as a writer on scientific and other topics for *Politiken*, the leading Danish newspaper. He is also known, under the pseudonym of Kumbel, as the author of numerous volumes of epigrammatical poems. These books have sold in the millions.

The game of Hex occurred to Hein while he was contemplating the famous four-color theorem of topology. (The theorem, proved in 1976, is that four colors are sufficient to make any map so that no two countries of the same color have a common boundary.) Hein introduced the game in 1942 with a lecture to students at the Institute. On December 26 of that year *Politiken* published an account of the game; it soon became enormously popular in Denmark under the name of Polygon. Pads on which the game could be played with a pencil were sold, and for many months *Politiken* ran a series of Polygon problems, with prizes for the best solutions.

In 1948, John F. Nash, then a graduate student in mathematics at Princeton University (later a professor at Massachusetts Institute of Technology and one of the nation's outstanding authorities on game theory), independently reinvented the game. It quickly captivated students of mathematics both at the Institute for Advanced Study and Princeton. The game was commonly called either Nash or John, the latter name referring mainly to the fact that it was often played on the hexagonal tiles of bathroom floors. It did not acquire the name Hex until 1952 when a version of the game was issued under that title by the firm of Parker Brothers, Inc.

Readers who would like to try Hex are advised to make mimeographed copies of the board. The game can be played on these sheets by marking the hexagons with circles and crosses. If you should prefer to play with removable pieces on a permanent board, a large one can easily be drawn on heavy cardboard or made by cementing together hexagonal tiles. If the tiles are big enough, ordinary checkers make convenient pieces.

One of the best ways to learn the subtleties of Hex is to play the game on a field with a small number of hexagons. When the game is played on a two-by-two board (four hexagons), the player who makes the first move obviously wins. On a three-by-three board the first player wins easily by making his first move in the center of the board [*see Fig. 34*]. Because "black" has a double play on both sides of his piece, there is no way in which his opponent can keep him from winning on his third move.

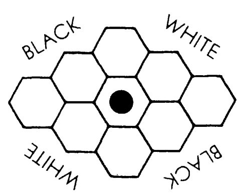

FIG. 34.

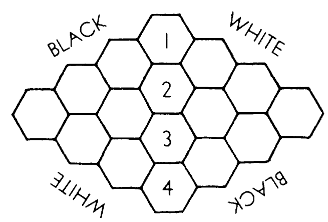

FIG. 35.

On a four-by-four board things begin to get complicated. The first player is sure to win if he immediately occupies any one of the four cells numbered in Figure 35. If he makes his opening play elsewhere, he can always be defeated. An opening play in cell 2 or 3 insures a win on the fifth move; an opening play in cell 1 or 4, a win on the sixth move.

On a five-by-five board it can still be shown that if the first player immediately occupies the hexagon in the center, he can win on his seventh move. On larger fields the analysis becomes enormously difficult. Of course the standard 11-by-11 board introduces such an astronomical number of complications that a complete analysis seems beyond the range of human computation.

Game theorists find Hex particularly interesting for the following reason. Although no "decision procedure" is known which will assure a win on a standard board, there is an elegant *reductio ad absurdum* "existence proof" that there is a winning strategy for the first player on a field of any size! (An existence proof merely proves the existence of something without telling you how to go about finding it.) The following is a highly condensed version of the proof (it can be formulated with much greater rigor) as it was worked out in 1949 by John Nash:

1. Either the first or second player must win, therefore there must be a winning strategy for either the first or second player.

2. Let us assume that the second player has a winning strategy.

3. The first player can now adopt the following defense. He first makes an arbitrary move. Thereafter he plays the winning second-player strategy assumed above. In short, he becomes the second player, but with an extra piece placed somewhere on the board. If in playing the strategy he is required to play on the cell where his first arbitrary move was made, he makes another arbitrary move. If later he is required to play where the second arbitrary move was made, he makes a third arbitrary move, and so on. In this way, he plays the winning strategy with one extra piece always on the field.

4. This extra piece cannot interfere with the first player's imitation of the winning strategy, for an extra piece is always an asset and never a handicap. Therefore the first player can win.

5. Since we have now contradicted our assumption that there is a winning strategy for the second player, we are forced to drop this assumption.

6. Consequently there must be a winning strategy for the first player.

There are a number of variations on the basic theme of Hex, including a version in which each player tries to force his opponent to make a chain. According to a clever proof devised by Robert Winder, a graduate student of mathematics at Princeton, the first player can always win this game on a board which has an even number of cells on a side, and the second player can always win on a board with an odd number.

After the reader has played Hex for a while, he may wish to tackle three problems devised by Hein. These are set forth in the three illustrations of Figure 36. The objective in all three problems is to find the first move that will insure a win for "white."

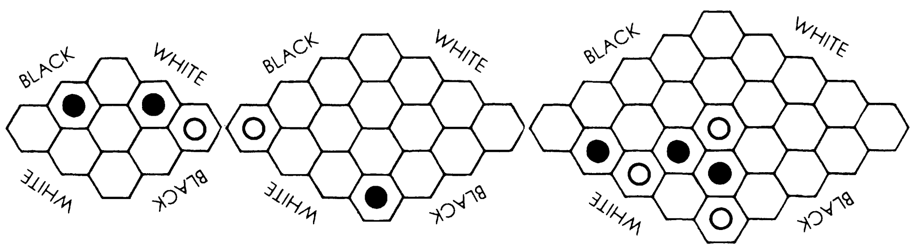

FIG. 36.
Three problems of Hex.

ADDENDUM

HEX can be played on several different types of fields which are topologically equivalent to the field of hexagons. A field of equilateral triangles, for example, may be used, placing the counters on the intersections. An ordinary checkerboard is isomorphic with a Hex field if one assumes that the squares connect diagonally in one direction only (say, NE and SW, but not NW and SE). Both boards seem to me less satisfying for actual play than the mosaic of hexagons.

Several shapes for a Hex field other than the diamond have been proposed. For example, Claude Shannon of the

Massachusetts Institute of Technology has suggested a field in the shape of an equilateral triangle. The winner is the first to complete a chain connecting all three sides of the triangle. Corner cells are regarded as belonging to both their adjacent sides. Nash's proof of first-player-win applies with equal force to this variant.

To counter the strong advantage held by the first player in the standard game of Hex, several proposals have been made. The first player may be forbidden to open on the short diagonal. The winner may be credited with how few moves it took him to win. The first player opens with one move, but thereafter each player has two moves per turn.

It is tempting to suppose that on an n by $n + 1$ board (e.g., a 10-by-11), with the first player taking the sides that are farthest apart, the relative advantages of the two players might be made more equal. Unfortunately, a simple strategy has been discovered which gives the second player a certain win. The strategy involves a reflection symmetry along a central axis. If you are the second player, you imagine the cells to be paired according to the scheme indicated by the letters in Figure 37. Whenever your opponent plays, you play on the other cell with the same letter. Owing to the shorter distance between your two sides of the board, it is impossible for you to lose!

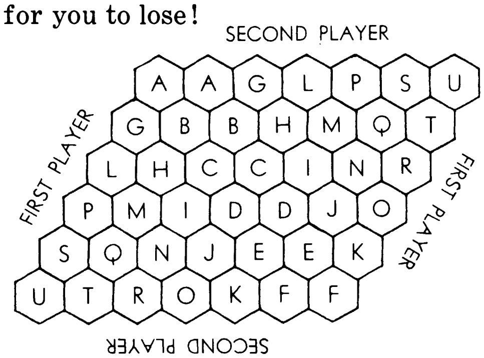

FIG. 37.
How second player pairs the cells to win on a "short" board.

A few words about general strategy in playing Hex. Quite a number of readers wrote that they were disappointed to discover that the first player has an easy win simply by taking the center cell, then extending a chain of adjacent cells toward his two sides of the board. They argued that since he always has a choice of two cells for the next link in the chain, it would be impossible to block him. Of course they failed to play long enough to discover that chains can be blocked by taking cells that are not adjacent to the ends of the chain. The game is much subtler than it first appears. Effective blocking often involves plays that seem to have no relationship to the chain that is being blocked.

A more sophisticated strategy is based on the following procedure. Play first in the center, then seek to form on each of your sides a chain of separated links that are either diagonal or vertical, like the two chains shown in Figure 38. If your opponent checks you vertically, you switch to a diagonal play and if he checks you diagonally, you switch to vertical. Of course, once you succeed in joining your two sides with a disconnected chain on which each missing link is a double play, you cannot be blocked. This is a good strat-

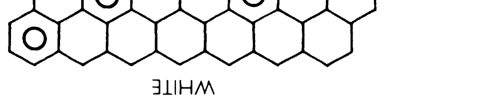

FIG. 38.

egy to play on novices, but it can be countered by proper defensive moves.

Still another strategy provided the basis of a Hex machine constructed by Claude Shannon and E. F. Moore, both at that time on the staff of Bell Telephone Laboratories. Here is Shannon's description of the device from his article on "Computers and Automata" in the *Proceedings of the Institute of Radio Engineers,* Vol. 41, October 1953:

After a study of this game, it was conjectured that a reasonably good move could be made by the following process. A two-dimensional potential field is set up corresponding to the playing board, with white pieces as positive charges and black pieces as negative charges. The top and bottom of the board are negative and the two sides positive. The move to be made corresponds to a certain specified saddle point in this field.

To test this strategy, an analog device was constructed, consisting of a resistance network and gadgetry to locate the saddle points. The general principle, with some improvements suggested by experience, proved to be reasonably sound. With first move, the machine won about seventy per cent of its games against human opponents. It frequently surprised its designers by choosing odd-looking moves which, on analysis, proved sound. We normally think of computers as expert at long, involved calculations and poor in generalized value judgments. Paradoxically, the positional judgment of this machine was good; its chief weakness was in end-game combinatorial play. It is also curious that the Hex-player reversed the usual computing procedure in that it solved a basically digital problem by an analog machine.

As a joke, Shannon also built a Hex machine which took the second move and always won, much to the puzzlement of

Hexperts who knew of the first player's strong advantage. The board was short in one direction (7 by 8), but mounted on a rectangular box in such a way that the inequality of sides was disguised. Few players were suspicious enough to count the cells along two edges. The machine, of course, played the winning reflection strategy previously described. It could have been constructed to respond instantly to moves, but thermistors were used to slow down its operation. It took one to eight seconds to reach a decision, thus conveying the impression that it was making a complicated analysis of the configuration on the field!

ANSWERS

SOLUTIONS to the three Hex problems given in Figure 36 are shown in Figure 39. A complete analysis of alternate lines of play is too lengthy to give; only the one correct first move for "white" is indicated by the crosses.

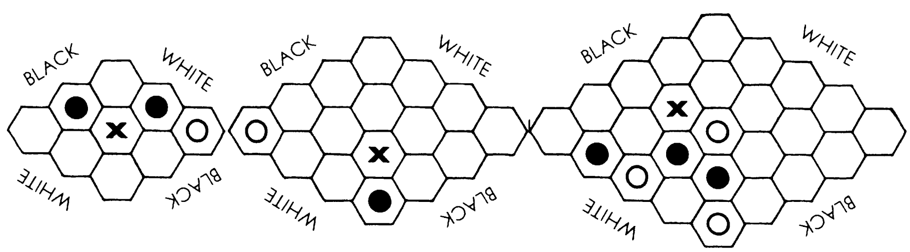

FIG. 39.

Several readers expressed a belief that in the third problem "white" could also win by playing on cell 22 (begin at the extreme left and number the rows up and to the right from 1 to 25). "Black," however, can defeat this by the following ingenious line of play:

White	*Black*
22	19
18	10
5	9
4	8
3	7

"White's" moves are forced in the sense that "black" has a quicker win unless "white" makes the indicated move. At the close of the above moves, "black" will have a chain with a double play at both ends and a double play to close one break within the chain, so there is no way "white" can prevent the win.