Geometry Final Exam

INSTRUCTIONS: This is a take-home exam. You may use both of my sets of course notes, *Journey Through Genius*, and your course notes, but no other source of information or assistance, human or non-human, though you may certainly contact me if you have questions. You should strictly avoid communicating anything at all in any form to the other course members, not even comments like "I have been having trouble with the first problem but otherwise I am nearly finished." Your answers are due noon, Monday, December 10, either uploaded to the Moodle site as a single file or emailed to lee@ms.uky.edu.

1. **Spherical Geometry.** In this problem we take a final dip into non-Euclidean geometry and develop an area formula in spherical geometry. Consider a sphere of radius r.

   (a) Prove that the area of a lune (two-sided polygon formed by two half great circles joining two antipodal points) on this sphere is $2ar^2$, where a is the measure (in radians) of the angle at which the two half great circles meet at either end.

   (b) Prove that the area of a spherical triangle T on the sphere is $r^2(a + b + c - \pi)$, where the measures (in radians) of the angles of the triangle are a, b, c. To do this, describe the triangle as the intersection of three hemispheres A, B, C, with the angles between them being the angles of the triangle itself. See the diagram for Exercise 3.67 of *Geometry for Middle School Teachers: Companion Problems for the Connected Mathematics Curriculum*, page 55. You may use the fact that the solution to Exercise 3.66 is $\text{area}(A \cup B \cup C) = \text{area}(A) + \text{area}(B) + \text{area}(C) - \text{area}(A \cap B) - \text{area}(A \cap C) - \text{area}(B \cap C) + \text{area}(A \cap B \cap C)$. In the diagram on page 55 the triangle T of interest is the intersection of three hemispheres. It is not so easy to see without a physical model, but the uncovered triangle T' on the back is congruent to T, and you may use this fact.

   (c) Explain why this shows that the sum of the measures of the angles of any spherical triangle exceeds π, and that this sum is not the same for all spherical triangles on a given sphere.

2. **Polyhedra.** In this problem you will prove something that those who construct geodesic domes and those who study Buckyballs know. Recall that you already proved for convex polyhedra that

$$2E = 3F_3 + 4F_4 + 5F_5 + \cdots. \tag{1}$$

You also know Euler's formula,

$$V - E + F = 2. \tag{2}$$

Assume that you have a polyhedron such that exactly three faces meet at each vertex, and every face is either a pentagon or a hexagon (though the faces need not be regular polygons). It turns out that there are infinitely many such polyhedra—two examples are the dodecahedron and the soccer ball.

(a) Prove that $3V = 2E$.

(b) Use this and Equations (1) and (2) to prove that there must be exactly 12 faces that are pentagons.

3. **Isometries.** In this problem we make a connection between complex numbers and formulas for isometries. You are already familiar with the fact that we can regard complex numbers $a + bi$ as points (a, b) in the plane. Consider a particular fixed complex number z on the unit circle centered at the origin. Define the function $f(w) = zw$ that maps the set of complex numbers into itself by multiplying each complex number w by z.

(a) Use what you proved about formulas for rotations in the plane to prove that f is a rotation.

(b) Use this connection between complex multiplication and rotations to find a complex number $z \neq 1$ such that $z^3 = 1$.

4. **Symmetry.** In this problem we examine some groups of symmetries of certain objects.

(a) Consider a square centered at the origin with vertices at the points $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$. The eight symmetries of the square are

- I (identity)
- R_{90} (counterclockwise rotation about the origin by 90 degrees)
- R_{180} (counterclockwise rotation about the origin by 180 degrees)
- R_{270} (counterclockwise rotation about the origin by 270 degrees)
- $F_{y=0}$ (reflection across the line $y = 0$)
- $F_{y=x}$ (reflection across the line $y = x$)
- $F_{x=0}$ (reflection across the $x = 0$)
- $F_{y=-x}$ (reflection across the line $y = -x$)

Make a "multiplication table" for the eight symmetries of the square. Use the above ordering of the symmetries to label the eight rows and the eight columns. For the row labeled by a symmetry X and the column labeled by a symmetry Y, the entry in that row and column should be the net result of *first* performing X and *second* performing Y. This composition is denoted $Y \circ X$.

Y

	$Y \circ X$	I	R_{90}	R_{180}	R_{270}	$F_{y=0}$	$F_{y=x}$	$F_{x=0}$	$F_{y=-x}$
	I								
	R_{90}								
X	R_{180}								
	R_{270}								
	$F_{y=0}$								
	$F_{y=x}$								
	$F_{x=0}$								
	$F_{y=-x}$								

(b) Using the appropriate subset of the above eight isometries, create a multiplication table for the symmetries of a non-square rectangle centered at the origin with vertices $(2, 1)$, $(-2, 1)$, $(-2, -1)$, and $(2, -1)$.

(c) Using the appropriate subset of the eight isometries in (4a), create a multiplication table for the symmetries of the polygon on the right of the diagram in Exercise 2.89 of *Geometry for Middle School Teachers: Companion Problems for the Connected Mathematics Curriculum*, page 31, assuming that it is centered at the origin.

(d) Your tables for (4b) and for (4c) should have the same number of symmetries, and both should be subtables of the table for the square—that is to say, each is a subgroup of the group of symmetries of the square. Are these two subgroups isomorphic—essentially identical in structure? Can the rows and columns be possibly reordered and the symmetries be relabeled, say, $A, B, C, \ldots$, so that the two tables become identical?

(e) Now make an addition table for the integers $0, 1, 2, 3$ modulo 4.

y

	$x + y$	0	1	2	3
	0				
x	1				
	2				
	3				

3

Is this table isomorphic to either or both of those in (4b) or (4c)? Can you explain why?

(f) Now make a multiplication table for the four complex numbers $1, -1, i, -i$.

$$y$$

xy	1	-1	i	$-i$
1				
-1				
i				
$-i$				

x

Is this table isomorphic to either or both of those in (4b) or (4c)? Can you explain why?