# Symmetries and Transformations

## 1 Informal Introduction to Symmetry

Examine the provided images.

1. What is meant by *symmetrical*? Write some definitions.
2. Are some of these "more symmetrical" than others?
3. Propose some methods of classification with respect to symmetry.

Helpful materials: transparencies, Miras, digital versions to paste into GeoGebra.

## 2 Introduction to Transformations

When exploring symmetry above, think about using transparencies to "reposition" the image on top of itself. This can be viewed as moving the entire plane onto itself in such a way that all distances are unchanged. The result of such an action is called an *isometry*, *rigid motion*, or *congruence transformation.* (What does the CCSSM call them?)

1. Classify the different types of such isometries that you have encountered. Write definitions.
2. Use GeoGebra or Sketchpad to illustrate these isometries, applying them to various figures.
3. Illustrate these isometries with "whole body" movement.

Note: The word "motion" might be misleading, since often we are considering isometry from the point of view of the final *result* of applying the isometry, and not the entire *movement* from start to finish. Also, sometimes a reflection is thought of as a flip or rotation through a third dimension, which leaves the plane during the process and may cause confusion between reflections and rotations. (It is also harder to visualize a reflection with respect to a plane in three dimensions as a "hyper-flip" or rotation through a fourth dimension, though this can be mathematically justified.)

# 3 Creating Symmetrical Figures

1. Use the notion of isometry to write a definition of symmetry.
2. Revisit your classification with respect to symmetry to use the language of isometries.
3. Construct figures with various kinds of symmetry using paper, GeoGebra, computer tools, iPad apps, people, etc. Polar graph paper can be helpful.
4. Solve Scott Kim's symmetry puzzles.
5. Go on a photographic "Symmetry Scavenger Hunt" to collect representatives of different symmetry types by taking pictures of objects you encounter around you.
6. Create a carefully designed symmetrical version of your name—first, last, or both.

# 4 Identifying Isometries

By now you have seen four types of isometries: translations, rotations, reflections, and glide reflections. The identity isometry can be regarded as a translation by a vector of length zero or a rotation by an angle of measure zero.

1. What elements do you need for each type of isometry in order to precisely specify its action?
2. For each type, precisely describe the relationship of a starting point $A$ to its image $B$ under the action of the isometry.
3. How much information do you need to identify a specific isometry? Is it enough to see the result of its action on a single point? On a line segment? On a circle? On a triangle? Experiment with GeoGebra or Geometer's Sketchpad.
4. Given a figure and its image under the action of an isometry, how can you precisely determine its elements? Practice with GeoGebra or Geometer's Sketchpad.

# 5 Coexisting Symmetries in Bounded Figures

You have seen some bounded figures with rotational symmetry only, and some with both rotational and reflectional symmetry.

1. Have you seen any with reflectional symmetry but no rotational symmetry?

2. Have you seen any with more than one line of reflectional symmetry but no rotational symmetry? Try to understand this result.

3. What is the net effect of carrying out a reflection in a line $\ell$ followed by a reflection in a line $m$? Consider various possible pairs of lines. Experiment with GeoGebra and Geometer's Sketchpad, with paper, and with people. Prove your result geometrically.

4. Prove your result algebraically. Note: First, you will need to spend some time figuring out formulas for reflections and rotations—you can concentrate on reflections across lines through the origin, and rotations about the origin.

5. How does this help you understand the classification of symmetry types of bounded figures?

# 6 Coexisting Symmetries in Strip Patterns

You have seen some strip patterns with translational symmetry. Think of these as restricted to a "strip" of the plane centered along the $x$-axis. These patterns have a "smallest" translational symmetry, of which all other translational symmetries are multiples or negative multiples.

1. Classify translations, rotations, reflections, and glide reflections of the plane that map this entire strip onto itself.

2. In each case develop algebraic formulas.

3. Determine the results of performing one of these strip isometries followed by another.

4. Use this to determine what possible combinations of symmetries can coexist for a given strip pattern.

5. Create your own examples, on paper or with software, for each of the symmetry types.

6. Find and photograph real-life examples of each of the possible symmetry types.

7. Find images of border patterns that arise in various cultures that represent each of the possible symmetry types.

# 7 Wallpaper Patterns

These patterns cover the entire plane. They have two "smallest" translational symmetries of which all other translational symmetries are positive or negative combinations.

1. Research the classification of the possible symmetries.
2. Practice using a flowchart to classify wallpaper patterns.
3. Find and photograph representatives of each of the possible symmetry types.
4. Create your own patterns on paper or with various software and apps.
5. Find images of wallpaper patterns that arise in various cultures that represent each of the possible symmetry types.
6. Study and present the explanation of why these patterns can only possibly contain 2-fold, 3-fold, 4-fold, and 6-fold rotational symmetry.
7. Research the mathematics connected to the work of M.C. Escher.