MA 514 Homework #2
Due Wednesday, September 5, in Class

1. Use Equation (1.1) to prove that if a graph G with n vertices has exactly $n-1$ edges and no isolated vertices, then it must have at least one vertex of degree 1.

2. Problem 1D.

3. Extend Problem 1H by finding and proving a combinatorial interpretation for the entries of the matrix A^ℓ, $\ell \geq 1$.

4. For any simple graph G, direct each edge arbitrarily to obtain a digraph G'. Construct a matrix B with rows indexed by the vertices of G', and columns indexed by the edges of G': entry b_{ve} equals -1 if vertex v is the tail of e, $+1$ if v is the head of e, and 0 otherwise. Find a combinatorial interpretation for the entries of BB^T.

5. Let G be a simple graph with vertex set $V = \{1, \ldots, n\}$. Assume that there is a positive weight w_{ij} associated with each edge $ij \in E(G)$. Define also $w_{ii} = 0$ for all i, and $w_{ij} = +\infty$ if $ij \notin E(G)$. Let W be the $n \times n$ matrix with entries w_{ij}. For any compatibly-shaped matrices A and B, define "weird" matrix multiplication by $C = AB$, where the product is given by

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}.$$

   (a) For any walk P in G, define the weight of the walk, $w(P)$, to be the sum of the weights of the edges in the the walk. For every pair of vertices x, y, prove that a minimum weight walk from x to y is necessarily a simple path.

   (b) For any integer $\ell \geq 1$, prove that entry ij of the matrix W^ℓ (using weird matrix multiplication) is the weight of a minimum weight walk from i to j using no more than ℓ edges.