MA 514 Homework #2

Due Monday, September 15, in Class

1. Extend Problem 1H by finding and proving a combinatorial interpretation for the entries of the matrix A^ℓ, $\ell \geq 1$.

2. For any simple graph G, direct each edge arbitrarily to obtain a digraph G'. Construct a matrix B with rows indexed by the vertices of G', and columns indexed by the edges of G': entry b_{ve} equals -1 if vertex v is the tail of e, $+1$ if v is the head of e, and 0 otherwise.

   (a) Find a combinatorial interpretation for the entries of BB^T.

   (b) Prove that a subset of rows of B is dependent (over $\mathbf{R}$) iff the corresponding set of vertices of G contains (but not necessarily equals) all of the vertices in some component of G. Suggestion: Think about a vector in the left nullspace of B as a certain way of labeling the vertices of G with real numbers.

   (c) Prove that the row rank of B equals $|V(G)| - 1$ iff G contains precisely one component.

   (d) Prove that a subset of columns of B is dependent iff the corresponding set of edges of G contains (but not necessarily equals) all of edges in some polygon of G.

   (e) Prove that the column rank of B equals $|E(G)|$ iff G contains no polygons.

   (f) Use the "Rank-Nullity" Theorem of Matrices to prove for a connected graph G that it is a tree iff it has $|V(G)| - 1$ edges.