MA515 Homework #3
Due Wednesday, September 14

1. A *tree* is a digraph that is acyclic (contains no cycles, directions of arcs unimportant) and is connected (every pair of vertices is joined by a path, directions of arcs unimportant). Prove that the following are equivalent for a digraph G with at least one edge. Try using some of the properties of the dimensions of the vector spaces associated with the vertex-edge incidence matrix A of G.

   (a) G is a tree.

   (b) G is minimally connected; i.e., G is connected, but no subgraph with the same vertex set and fewer edges is connected.

   (c) G is maximally acyclic; i.e, G is acyclic, but no supergraph with the same vertex set and more edges is acyclic.

   (d) $|V(G)| = |E(G)| + 1$ and G is connected.

   (e) $|V(G)| = |E(G)| + 1$ and G is acyclic.

2. Let A be the vertex-edge incidence matrix of a digraph G with at least one edge. Let M be any square submatrix of A, determined by selecting any sets of equal numbers of rows and columns of A, not necessarily adjacent. Prove that the determinant of M is 0, 1, or -1. Suggestion: Recall how to calculate a determinant by expansion along a column.

3. Exercise 2.14 of my notes, parts (1), (2), (3) and (6). It might be helpful to think of the vectors in E as columns of an $n \times r$ matrix.