INSTRUCTIONS: This is a take-home exam. You may use my course notes, the course text, and your course notes, but no other source of information or assistance, human or non-human. You may certainly see me if you have questions. Your answers are due Wednesday, November 1, at 4:00 pm — give them directly to me or slide them under my office door.

1. Let $P \subset \mathbb{R}^n$ be a polytope. Suppose $V = \{v^1, \ldots, v^m\}$ is a finite subset of $P$ such that for every $c \in \mathbb{R}^n$ there is at least one point in $V$ at which the function $c^T x$ attains its maximum value. Prove that every point $p$ in $P$ is a convex combination of the points in $V$ by applying a theorem of the alternatives to the following system:

$$
\begin{bmatrix}
    v^1 & \cdots & v^m \\
    1 & \cdots & 1
\end{bmatrix}
\lambda =
\begin{bmatrix}
    p \\
    1
\end{bmatrix}
\lambda \geq O
$$

2. Assume that you have used the simplex method to find an optimal solution to the linear program

$$
\begin{aligned}
\max & c^T x \\
\text{s.t.} & Ax = b \\
& x \geq O
\end{aligned}
$$

Assume that $A$ has full row rank and that $\overline{y}^T = c_B^T A_B^{-1}$ is associated with the final optimal basis $B$. Prove that $\overline{y}$ is optimal for the dual $(D)$ of $(P)$:

$$
\begin{aligned}
\min & y^T b \\
\text{s.t.} & y^T A \geq c^T
\end{aligned}
$$

Suggestion: Show that $\overline{y}$ is feasible for $(D)$ and satisfies complementary slackness.

3. Consider the linear programs $(P)$ and $(P(u))$:

$$
\begin{aligned}
\max & c^T x & \max & c^T x \\
\text{s.t.} & Ax = b & \text{s.t.} & Ax = b + u \\
x \geq O & & x \geq O
\end{aligned}
$$

$(P)$ $(P(u))$

Assume that $(P)$ has an optimal objective function value $z^*$. Suppose that there exists a vector $y^*$ and a positive real number $\varepsilon$ such that the optimal objective function value
$z^*(u)$ of $(P(u))$ equals $z^* + u^T y^*$ whenever $\|u\| < \varepsilon$. Prove that $y^*$ is an optimal solution to the dual of $(P)$. Suggestion: Let $\overline{y}$ be optimal for the dual of $(P)$ and use a previous homework problem (Exercise 10.7) to (eventually) show $\overline{y} = y^*$.

4. An affine combination of points $v^1, \ldots, v^m \in \mathbb{R}^n$ is a linear combination

$$\lambda_1 v^1 + \cdots + \lambda_m v^m$$

for which we also require

$$\lambda_1 + \cdots + \lambda_m = 1.$$ 

Let $S = \{v^1, \ldots, v^m\}$ be a subset of $\mathbb{R}^n$. Define $S$ to be affinely independent if there is no linear combination

$$\lambda_1 v^1 + \cdots + \lambda_m v^m = O$$

for which

$$\lambda_1 + \cdots + \lambda_m = 0$$

other than $\lambda_1 = \cdots = \lambda_m = 0$.

(a) Prove that the set $S$ is affinely independent if and only if there is no element of $S$ that can be written as an affine combination of the remaining elements of $S$.

(b) Prove that the set $S$ is affinely independent if and only if the following subset of $\mathbb{R}^{n+1}$ is linearly independent:

$$\left\{ \begin{bmatrix} v^1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} v^m \\ 1 \end{bmatrix} \right\}$$

(c) Prove that the set $S$ is affinely independent if and only if the set

$$\{ v^1 - v^m, \ldots, v^{m-1} - v^m \} \subset \mathbb{R}^n$$

is linearly independent.

5. Exercise (Linear over $\text{GF}(2) \not\leftrightarrow \text{graphic}$) on the top of page 54 of the text. That is to say, prove that there is no graph $G$ such that the graphic matroid associated with $G$ is isomorphic to the Fano matroid. Suggestion: Try explicitly and systematically to construct a graph $G$ with 7 edges having the same independent sets as the Fano matroid.