

TWO COIN MORRA

This game is played by two players, R and C . Each player hides either one or two silver dollars in his/her hand. Simultaneously, each player guesses how many coins the other player is holding. If R guesses correctly and C does not, then C pays R an amount of money equal to the *total* number of dollars concealed by *both* players. If C guesses correctly and R does not, then R pays C an amount of money equal to the *total* number of dollars concealed by *both* players. If both players guess correctly or incorrectly, no money exchanges hands.

Clearly, each player must decide how many coins to hide and what number to guess. We will use the notation (1,2) to mean that a player hides 1 dollar and guesses “2.” We can represent the possible outcomes of a round of play by a *payoff* matrix, indicating how much R will receive from C given the strategies followed by each player. A negative number means that C receives money from R .

		C			
		(1,1)	(1,2)	(2,1)	(2,2)
R	(1,1)	0	2	-3	0
	(1,2)	-2	0	0	3
	(2,1)	3	0	0	-4
	(2,2)	0	-3	4	0

This is an example of a *finite two-person zero-sum game*. “Finite” refers to the fact that each player has a finite number of strategies. “Zero-sum” refers to the fact that what one player gains in wealth, the other loses.

Suppose R decides to use only strategy (1,2). This is an example of a *pure strategy*. Since the minimum entry in that row is -2 , R can guarantee losing no more than \$2 per round, and this will happen if C consistently uses strategy (1,1). But what if R decides to use either strategy (1,2) or (2,1), each half of the time, but randomly mixed so that there is no detectable pattern? For example, he could flip a coin to decide which of the two strategies to use. This is an example of a *mixed strategy*. To calculate the expected outcome, we mix the second and third rows by multiplying them each by $1/2$ and adding them together. The result is

$$\begin{aligned}
 & \frac{1}{2} \begin{bmatrix} -2 & 0 & 0 & 3 \end{bmatrix} \\
 & \quad + \\
 & \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 & -4 \end{bmatrix} \\
 & \quad \parallel \\
 & \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{bmatrix}.
 \end{aligned}$$

Since the minimum entry is $-1/2$, R can expect to lose no more than half a dollar per round on the average, and he can expect this to happen if C consistently chooses strategy (2,2).

Can R do better than this? If R uses each strategy $1/4$ of the time, then R mixes his rows by multiplying each of them by $1/4$ and adding them together, giving

$$\begin{aligned} & \frac{1}{4} \begin{bmatrix} 0 & 2 & -3 & 0 \end{bmatrix} \\ & \quad + \\ & \frac{1}{4} \begin{bmatrix} -2 & 0 & 0 & 3 \end{bmatrix} \\ & \quad + \\ & \frac{1}{4} \begin{bmatrix} 3 & 0 & 0 & -4 \end{bmatrix} \\ & \quad + \\ & \frac{1}{4} \begin{bmatrix} 0 & -3 & 4 & 0 \end{bmatrix} \\ & \quad \parallel \\ & \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}. \end{aligned}$$

Since the minimum entry is $-1/4$, R would expect to lose no more than a quarter dollar per round on the average, and he can expect this to happen if C consistently uses only strategies (1,2) and (2,2).

Problem: Try to find a mixed strategy for R which is even better. In particular, try to find nonnegative numbers p_1, p_2, p_3, p_4 that sum to one such that the minimum entry in

$$\begin{aligned} & p_1 \begin{bmatrix} 0 & 2 & -3 & 0 \end{bmatrix} \\ & \quad + \\ & p_2 \begin{bmatrix} -2 & 0 & 0 & 3 \end{bmatrix} \\ & \quad + \\ & p_3 \begin{bmatrix} 3 & 0 & 0 & -4 \end{bmatrix} \\ & \quad + \\ & p_4 \begin{bmatrix} 0 & -3 & 4 & 0 \end{bmatrix} \end{aligned}$$

is at least zero. Can you find a mixture where the minimum entry is larger than zero?

C 's strategies can be studied in a similar manner. For example, suppose C decides to use only strategy (2,2). The numbers in the matrix represent amounts that C pays R , so C is interested in looking at the maximum entry to see how badly he will do. Since the maximum entry in column 4 is 3, C can expect to lose at most \$3 per round, and this happens if R consistently uses strategy (1,2). If C uses each of his strategies a fourth of the time, we must

mix the columns by multiplying each of them by $1/4$ and adding them together. This yields

$$\frac{1}{4} \begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -3 \\ 0 \\ 0 \\ 4 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 \\ 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}.$$

Since the maximum entry is $1/4$, by using this mixed strategy, C can expect to lose no more than a quarter dollar per round on the average, and he can expect this if R consistently uses only strategies (1,2) and (2,2).

Of course, by symmetry, we can expect the best strategy for R to be the same as the best strategy for C .

Here is how to find the best strategy for R by linear programming: Find the optimal solution to

$$\begin{aligned} & \max z \\ & \text{s.t. } p_1 + p_2 + p_3 + p_4 = 1 \\ & z \leq 0p_1 - 2p_2 + 3p_3 + 0p_4 \\ & z \leq 2p_1 + 0p_2 + 0p_3 - 3p_4 \\ & z \leq -3p_1 + 0p_2 + 0p_3 + 4p_4 \\ & z \leq 0p_1 + 3p_2 - 4p_3 + 0p_4 \\ & p_1, p_2, p_3, p_4 \geq 0 \end{aligned}$$

One (the only?) optimal solution is $p_2 = 3/5$, $p_3 = 2/5$, with a value of 0. By symmetry, this will also be C 's optimal strategy.

Problem: What if R and C do not announce their guesses simultaneously, but R guesses first, C guesses second, and then both open their hands to see who is correct? Does the fact that R guesses before C give C any advantage, even though C still does not know how many coins R is hiding? Try solving this problem using linear programming.