MA614 Homework #3

Due Wednesday, January 31

1. Define $(c_1, \ldots, c_k)$ to be a *weak k-composition* of n if $c_1, \ldots, c_k$ are nonnegative integers that sum to n (i.e., we are now allowed to use 0 as a part).

   (a) Find a bijection between (ordinary) k-compositions of $n + k$ and weak k-compositions of n.

   (b) Find a formula for the number of weak k-compositions of n.

2. You have already proved, using generating functions, that the number of compositions $(c_1, \ldots, c_k)$ of n in which c_i comes in $2^{c_i - 1}$ "colors" equals 3^{n-1}, $n \geq 1$. Find a bijective proof of this fact by constructing a bijection between such colored compositions and sequences $d_1 \cdots d_{n-1}$ in which each $d_j \in \{0, 1, 2\}$. Suggestion: Start with such a sequence and interpret the 2's as plus signs.

3. Let a_n be the number of compositions $(c_1, \ldots, c_k)$ of n in which the parts $c_1, c_2, \ldots, c_k$ are alternately odd and even, starting with c_1 being odd. By convention, let's take $a_0 = 1$, since all parts of the empty composition (of which there are none) vacuously satisfy the condition. Find the ogf for $(a_n)_{n \geq 0}$. Suggestion: First write out the necessary generating function "long-hand" in the form

$$1 + (\cdots) + (\cdots)(\cdots) + (\cdots)(\cdots)(\cdots) + (\cdots)(\cdots)(\cdots)(\cdots) + \cdots,$$

   where the expressions in the parentheses need not all be identical.

4. Use the formula for the number of derangements D_n of length n to prove the recursions:

   (a) $D_n = nD_{n-1} + (-1)^n$.

   (b) $D_n = (n - 1)(D_{n-1} + D_{n-2})$.

5. Let's consider lattice paths from $(0, 0, 0)$ to (p, q, r) in the three-dimensional lattice $\mathbf{Z}^3$, in which the allowable steps are called "X" (add $(1, 0, 0)$), "Y" (add $(0, 1, 0)$) and "Z" (add $(0, 0, 1)$). Determine a formula $L(p, q, r)$ for the number of such paths.