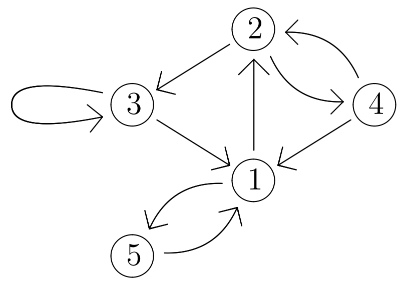
MA614 Homework #4

Due Wednesday, February 7

1. Suppose $f(x) = \sum_{n\geq 0} a_n x^n$. Find an expression for $\sum_{n\geq 0} a_{2n} x^n$ in terms of $f(x)$.

2. Let G be the following directed graph.



Let c_n be the number of closed directed walks with n edges starting and ending at the vertex 1. Recall that a closed directed walk is allowed to repeat vertices and edges. For instance, the directed walk $1 \to 5 \to 1 \to 2 \to 3 \to 3 \to 1$ is a closed directed walk from 1 to itself of length 6. Find the ogf $\sum_{n\geq 0} c_n x^n$. Suggestion: What are the "simplest" closed directed walks from 1 to 1; i.e., those directed walks that cannot be composed by concatenating two other directed closed walks from 1 to 1?

3.

   (a) Determine the number of compositions of d of the form $(1, \ldots, 1, 2, \ldots, 2)$ such that the composition consists of a nonempty sequence of 1's followed by a nonempty sequence of 2's, $d \geq 0$.

   (b) Find a bijection between the compositions $(c_1, \ldots, c_k)$ of n in which c_i comes in $\lfloor \frac{c_i - 1}{2} \rfloor$ "colors" ($\lfloor \cdot \rfloor$ denotes the integer round-down function), and the compositions of $n - 3$ into 1's and 2's.

4. Let $c_{m,n}$ be the number of lattice paths from $(0,0)$ to (m,n) using only steps of the form $E = (1,0)$, $N = (0,1)$, and $D = (1,1)$.

   (a) Find the ogf $\sum_{m\geq 0} \sum_{n\geq 0} c_{m,n} x^m y^n$.

   (b) Prove that

$$\frac{x^j}{(1-x)^{j+1}} = \sum_{n\geq 0} \binom{n}{j} x^n.$$

(c) Prove using generating functions that

$$c_{m,n} = \sum_{j \geq 0} 2^j \binom{m}{j} \binom{n}{j}.$$

Suggestion: Multiply both sides by $x^m y^n$, sum over m and n, and work toward the ogf you obtained in part (a).

5. Here is another way to find the formula for $C(n,n)$, the number of lattice paths from $(0,0)$ to (n,n) not crossing above the line $y = x$ using only steps of the form $E = (1,0)$ and $N = (0,1)$. For convenience, let's write $c_n = C(n,n)$, $n \geq 0$. Note that $c_0 = 1$.

(a) Prove the recurrence relation $c_n = c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0$, $n \geq 1$. Suggestion: For any valid path P let (p,p) be the first time after $(0,0)$ that P touches the line $y = x$ again. The portion of P from $(0,0)$ to (p,p) begins by moving E to $(1,0)$, follows some path Q to $(p, p-1)$, and ends with N to (p,p). Now think about Q's relation to the line $y = x - 1$.

(b) Let $f(x) = \sum_{n \geq 0} c_n x^n$ be the ogf for the sequence $(c_n)_{n \geq 0}$. Use the recurrence relation to find $f(x)$.

(c) Now use a Taylor series expansion or the Binomial Theorem to verify that $c_n = \frac{1}{n+1}\binom{2n}{n}$.