MA614 Homework #5

Due Monday, February 19

Reminder: Exam #1 will be an in-class exam given on Wednesday, February 14, with the material up to and including the discussion of the unsigned Stirling numbers of the first kind, $c(n,k)$.

1. Give a generating function proof of the following identity, where n is a positive integer:

$$\sum_{i+j+k=n;\ i,j,k\geq 1} \binom{n}{i}\binom{n}{j}\binom{n}{k} = \binom{3n}{n} - 3\binom{2n}{n} + 3.$$

2. Give a combinatorial (non-generating function) proof of the identity

$$\binom{\binom{n}{2}}{2} = 3\binom{n}{4} + n\binom{n-1}{2}.$$

3. Consider a lattice path P from $(0,0)$ to (m,n) using only steps of the form $E=(1,0)$, $N=(0,1)$, and $D=(1,1)$. Define the weight of P to be $w(P)=(-1)^k$ where k is the number of steps of type D in P. Let $a_{mn}=\sum_P w(P)$ where the sum is taken over all lattice paths P of the above form from $(0,0)$ to (m,n).
   (a) Find the ogf $f(x,y)=\sum_{m,n\geq 0} a_{mn}x^m y^n$.
   (b) Use the ogf to find an explicit formula for a_{mn}.
   (c) Ignore the ogf and directly use induction to verify the formula you found in (b).

4. Consider the set $\mathcal{A}_n$ of lattice paths from $(0,0)$ to $(n,0)$ that do not cross below the x-axis, using only steps of the form $U=(1,1)$, $D=(1,-1)$ and $H=(1,0)$. Define the weight of a path P to be $w(P)=2^k$ where k is the number of steps of type H in P. Let $a_n=\sum_P w(P)$ where the sum is taken over all lattice paths $P\in\mathcal{A}_n$.
   (a) Find the ogf $\sum_{n\geq 0} a_n x^n$.
   (b) Using the ogf found in (a), prove that a_n is the Catalan number $c_{n+1}=\frac{1}{n+2}\binom{2n+2}{n+1}$.

5. Directly count the following:
   (a) The number of $\pi\in S_n$ with exactly $n-1$ cycles.
   (b) The number of $\pi\in S_n$ with exactly $n-2$ cycles.
   (c) The number of $\pi\in S_n$ with exactly $n-3$ cycles.