MA614 Homework #6

Due Monday, February 26

1. Give a combinatorial proof of the following identity, where n is a positive integer:

$$\sum_{i+j+k=n;\ i,j,k\geq 1} \binom{n}{i}\binom{n}{j}\binom{n}{k} = \binom{3n}{n} - 3\binom{2n}{n} + 3.$$

2. Prove (using the definitions and algebra) that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \delta_{0,k}, & \text{if } n = 0, \\ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}, & \text{if } n > 0. \end{cases}$$

3. Prove that, for $n \geq 1$,

$$\prod_{k=0}^{n-1} (1 + q^k x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

4. Prove that

$$S(n,k) = \begin{cases} \delta_{n,k}, & \text{if } n = 0, \\ S(n-1,k-1) + kS(n-1,k), & \text{if } n > 0. \end{cases}$$

5. Prove that

$$\sum_{n\geq 0} S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$