MA614 Homework #8

Due Wednesday, March 21

1. Find the egf for:
   (a) The number of permutations π of an n-set such that $\pi^4 = id$.
   (b) The number of partitions of an n-set such that one element from each block is marked.
   (c) The number of permutations of an n-set where each k-cycle has one of 2^k colors.
   (d) The number of ways to partition an n-set and then color each block with one of two colors.

2. A sequence $v(n)$ is defined to have the non-zero entries $v(0) = 1$, $v(2) = 4$, $v(4) = 24$, and $v(n) = 0$ for all other n. For a labeled rooted tree T define its weight $w(T)$ by the product
$$w(T) = \prod_{x \in T} v(\text{number of children of the node } x).$$
   Find an explicit expression for a_n, given by
$$a_n = \sum_T w(T),$$
   where the sum is over all labeled rooted trees T on n nodes.

3. Define $h(n)$ by the sum
$$h(n) = \sum_F (-1)^{\text{number of components of } F},$$
   where the sum is over all labeled rooted forests F with the vertices labeled by the set $[n]$. Prove that $h(n) = -(n-1)^{n-1}$. (Note: A labeled rooted forest is a collection of labeled rooted trees where no two vertices in the forest have the same label.) Suggestion: You are likely to find the more general version (for arbitrary integer k) of the Lagrange inversion formula helpful.

4. Let $f(m, n)$ be the number of $m \times n$ matrices with entries from $\{0, 1\}$ with at least one 1 in every row and column. Use the principle of inclusion/exclusion to show
$$f(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^m.$$

5. There are n husband and wife couples (hence $2n$ people) that are to be seated around a circular table with $2n$ chairs. Prove that the number of ways of seating the $2n$ people around the table so that no one sits next to his or her spouse is

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}2^i(2n-i-1)!.$$

Note: There is no requirement that the sexes must alternate in the seating.