MA614 Homework #9

Due Wednesday, March 28

1. For any permutation $\pi = a_1 \cdots a_n$ of $[n]$, recall that $\mathrm{inv}(\pi)$, the number of inversions of π, is the number of pairs $i < j$ such that $a_i > a_j$. Let the sign of π equal $(-1)^{\mathrm{inv}(\pi)}$.

   (a) Fix any $k \in [n-1]$. Prove that interchanging a_k and a_{k+1} in a permutation π of $[n]$ is a sign-reversing involution on S_n.

   (b) Fix any $j, k \in [n]$, $j < k$. Prove that interchanging a_j and a_k n a permutation π of $[n]$ is a sign-reversing involution on S_n.

2. Use a sign-reversing involution to prove that

$$\begin{bmatrix} 2n \\ 2k \end{bmatrix}_{q=-1} = \binom{n}{k}.$$

3. Prove the following equality using a sign-reversing involution:

$$\sum_{i=k}^{n} S(n,i)s(i,k) = 0 \text{ for } k < n.$$

4. Extra Credit. Use problem #1 above to find a 3×3 determinantal expression for the number of ordered triples (P_1, P_2, P_3) of non-intersecting lattice paths using steps of $E = (1,0)$ and $N = (0,1)$, where

$$\begin{aligned} P_1 &: (2,0) \to (k+2, n-k) \\ P_2 &: (1,1) \to (k+1, n-k+1) \\ P_3 &: (0,2) \to (k, n-k+2) \end{aligned}$$

   This generalizes the result we saw in class involving two paths.