Sign Reversing Involutions and Lattice Paths

Because I was covering the material a little quickly on Friday I decided to write up some notes on the lattice paths.

Let $u_1 = (0,1)$, $v_1 = (k, n-k+1)$, $u_2 = (1,0)$, and $v_2 = (k+1, n-k)$. The problem is to count the number of ordered pairs (P_1, P_2) of lattice paths, using only the steps E and N, for which P_1 goes from u_1 to v_1, P_2 goes from u_2 to v_2, and P_1 and P_2 do not intersect. Denote this set of paths by F. (Draw pictures for yourself.)

We begin by considering a larger set of ordered pairs of lattice paths: $S = \{(P_1, P_2) : P_1$ goes from u_1 to v_i and P_2 goes from u_2 to v_j for $i \neq j\}$. We sign the set S by $S^+ = \{(P_1, P_2) : P_1 : u_1 \to v_1$ and $P_2 : u_2 \to v_2\}$, and $S^- = \{(P_1, P_2) : P_1 : u_1 \to v_2$ and $P_2 : u_2 \to v_1\}$. So $\epsilon(P_1, P_2) = +1$ if $(P_1, P_2) \in S^+$ and $\epsilon(P_1, P_2) = -1$ if $(P_1, P_2) \in S^-$.

Now define an involution π on S as follows: If (P_1, P_2) do not intersect then set $\pi(P_1, P_2) = (P_1, P_2)$ (a fixed point). If (P_1, P_2) intersect then let q be the first lattice point of intersection as you follow the paths. Set $\pi(P_1, P_2) = (P_1', P_2')$ where P_1' is obtained by following P_1 from u_1 to q and then following P_2 from q to the end of P_2, and P_2' is obtained by following P_2 from u_2 to q and then following P_1 from q to the end of P_1. (Draw pictures!)

We now make the following observations:

1. $F \subset S^+$.

2. All pairs of paths in $S \setminus F$ must intersect (in fact, paths in S^- must cross), so F is the set of fixed points of π.

3. The number of paths $P : u_1 \to v_1$ is the same as the number of paths from $(0,0)$ to $(k, n-k)$, namely, $\binom{n}{k}$.

4. Similarly, the number of paths $P : u_2 \to v_2$ is the same as the number of paths from $(0,0)$ to $(k, n-k)$, namely, $\binom{n}{k}$.

5. The number of paths $P : u_1 \to v_2$ is the same as the number of paths from $(0,0)$ to $(k+1, n-k-1)$, namely, $\binom{n}{k+1}$.

6. The number of paths $P : u_2 \to v_1$ is the same as the number of paths from $(0,0)$ to $(k-1, n-k+1)$, namely, $\binom{n}{k-1}$.

7. Thus the number of pairs of paths in $S^+ = \binom{n}{k}\binom{n}{k} = \binom{n}{k}^2$.

8. Similarly the number of pairs of paths in $S^- = \binom{n}{k+1}\binom{n}{k-1}$.

Therefore

$$|F| = \sum_{a \in S} \epsilon(a)$$

yields

$$|F| = |S^+| - |S^-| = \binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1},$$

or

$$|F| = \begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} \\ \\ \binom{n}{k-1} & \binom{n}{k} \end{vmatrix}$$

As a consequence, $\binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1} \geq 0$ and so $\binom{n}{k+1}\binom{n}{k-1} \leq \binom{n}{k}^2$, and thus the sequence $\binom{n}{k}$, $k = 0, \ldots, n$ is confirmed to be log-concave.