Polytopes Course Notes

Carl W. Lee
Department of Mathematics
University of Kentucky
Lexington, KY 40506
lee@ms.uky.edu

Fall 2013

Contents

1 Polytopes

Two excellent references are [16] and [51].

1.1 Convex Combinations and V-Polytopes

Definition 1.1 Let $v^1, \ldots, v^m$ be a finite set of points in $\mathbf{R}^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbf{R}$. Then

$$\sum_{j=1}^{m} \lambda_j v^j$$

is called a *linear combination* of $v^1, \ldots, v^m$. If $\lambda_1, \ldots, \lambda_m \geq 0$ then it is called a *nonnegative* (or *conical*) *combination*. If $\lambda_1 + \cdots + \lambda_m = 1$ then it is called an *affine combination*. If both $\lambda_1, \ldots, \lambda_m \geq 0$ and $\lambda_1 + \cdots + \lambda_m = 1$ then it is called *convex combination*. Note: We will regard an empty linear or nonnegative combination as equal to the point O, but will not consider empty affine or convex combinations.

Exercise 1.2 Give some examples of linear, nonnegative, affine, and convex combinations in $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$. Include diagrams. $\square$

Definition 1.3 The set $\{v^1, \ldots, v^m\} \subset \mathbf{R}^n$ is *linearly independent* if the only solution to $\sum_{j=1}^{m} \lambda_j v^j = O$ is the trivial one: $\lambda_j = 0$ for all j. Otherwise the set is *linearly dependent*.

Exercise 1.4 Prove that the set $S = \{v^1, \ldots, v^m\} \subset \mathbf{R}^n$ is linearly dependent if and only if there exists k such that v^k can be written as a linear combination of the elements of $S \setminus \{v^k\}$. $\square$

Solution: Assume S is linearly dependent. Then there is a nontrivial solution to $\sum_{j=1}^{m} \lambda_j v^j = O$. So there exists k such that $\lambda_k \neq 0$. Then

$$v^k = \sum_{j \neq k} \frac{-\lambda_j}{\lambda_k} v^j.$$

Therefore v^k can be written as a linear combination of the elements of $S \setminus \{v^k\}$.

Conversely, suppose there is a k such that v^k can be written as a linear combination of the elements of $S \setminus \{v^k\}$, say $v^k = \sum_{j \neq k} \lambda_j v^j$. Set $\lambda_k = -1$. Then $\sum_{j=1}^{m} \lambda_j v^j = O$ provides a nontrivial linear combination of the elements of S equaling O. Therefore S is linearly dependent.

Definition 1.5 The set $\{v^1, \ldots, v^m\} \subset \mathbf{R}^n$ is *affinely independent* if the only solution to $\sum_{j=1}^m \lambda_j v^j = O$, $\sum_{j=1}^m \lambda_j = 0$, is the trivial one: $\lambda_j = 0$ for all j. Otherwise the set is *affinely dependent*.

Exercise 1.6 Prove that the set $S = \{v^1, \ldots, v^m\} \subset \mathbf{R}^n$ is affinely dependent if and only if there exists k such that v^k can be written as an affine combination of the elements of $S \setminus \{v^k\}$. $\square$

Solution: Assume S is affinely dependent. Then there is a nontrivial solution to $\sum_{j=1}^m \lambda_j v^j = O$, $\sum_{j=1}^m \lambda_j = 0$. So there exists k such that $\lambda_k \neq 0$. Then

$$v^k = \sum_{j \neq k} \frac{-\lambda_j}{\lambda_k} v^j.$$

Note that

$$\begin{array}{rcl} \sum_{j \neq k} \frac{-\lambda_j}{\lambda_k} & = & \frac{1}{\lambda_k}(-\sum_{j \neq k} \lambda_j) \\ & = & \frac{1}{\lambda_k} \lambda_k \\ & = & 1. \end{array}$$

Therefore v^k can be written as an affine combination of the elements of $S \setminus \{v^k\}$.

Conversely, suppose there is a k such that v^k can be written as an affine combination of the elements of $S \setminus \{v^k\}$, say $v^k = \sum_{j \neq k} \lambda_j v^j$, where $\sum_{j \neq k} \lambda_j = 1$. Set $\lambda_k = -1$. Then $\sum_{j=1}^m \lambda_j v^j = O$ provides a nontrivial affine combination of the elements of S equaling O, since $\sum_{j=1}^m \lambda_j = 0$.

Exercise 1.7 Prove that the set $\{v^1, \ldots, v^m\}$ is affinely independent if and only if the set $\{v^1 - v^m, v^2 - v^m, \ldots, v^{m-1} - v^m\}$ is linearly independent. $\square$

Solution: Assume that the set $\{v^1, \ldots, v^m\}$ is affinely independent. Assume that $\sum_{j=1}^{m-1} \lambda_j (v^j - v^m) = O$. We need to prove that $\lambda_j = 0$, $j = 1, \ldots, m-1$. Now $\sum_{j=1}^{m-1} \lambda_j v^j - (\sum_{j=1}^{m-1} \lambda_j) v^m = O$. Set $\lambda_m = -\sum_{j=1}^{m-1} \lambda_j$. Then $\sum_{j=1}^m \lambda_j v^j = O$ and $\sum_{j=1}^m \lambda_j = 0$. Because the set $\{v^1, \ldots, v^m\}$ is affinely independent, we deduce $\lambda_j = 0$, $j = 1, \ldots, m$. Therefore the set $\{v^1 - v^m, \ldots, v^{m-1} - v^m\}$ is linearly independent.

Conversely, assume that the set $\{v^1 - v^m, \ldots, v^{m-1} - v^m\}$ is linearly independent. Assume that $\sum_{j=1}^m \lambda_j v^j = O$ and $\sum_{j=1}^m \lambda_j = 0$. We need to prove that $\lambda_j = 0$, $j = 1, \ldots, m$. Now $\lambda_m = -\sum_{j=1}^{m-1} \lambda_j$, so $\sum_{j=1}^{m-1} \lambda_j v^j - (\sum_{j=1}^{m-1} \lambda_j) v^m = O$. This is equivalent to $\sum_{j=1}^{m-1} \lambda_j (v^j - v^m) = O$. Because the set $\{v^1 - v^m, \ldots, v^{m-1} - v^m\}$ is linear independent, we deduce $\lambda_j = 0$, $j = 1, \ldots, m-1$. But $\lambda_m = -\sum_{j=1}^{m-1} \lambda_j$, so $\lambda_m = 0$ as well. Therefore the set $\{v^1, \ldots, v^m\}$ is affinely independent.

Definition 1.8 A subset $S \subseteq \mathbf{R}^n$ is a *subspace* (respectively, *cone, affine set, convex set*) if it is closed under all linear (respectively, nonnegative, affine, convex) combinations of its elements. Note: This implies that subspaces and cones must contain the point O.

Exercise 1.9 Give some examples of subspaces, cones, affine sets, and convex sets in $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$. $\square$

Exercise 1.10 Is the empty set a subspace, a cone, an affine set, a convex set? Is $\mathbf{R}^n$ a subspace, a cone, an affine set, a convex set? $\square$

Exercise 1.11 Prove that a nonempty subset $S \subseteq \mathbf{R}^n$ is affine if and only if it is a set of the form $L + x$, where L is a subspace and $x \in \mathbf{R}^n$. $\square$

Exercise 1.12 Are the following sets subspaces, cones, affine sets, convex sets?

1. $\{x \in \mathbf{R}^n : Ax = O\}$ for a given matrix A.

   Solution: This set is a subspace, hence also a cone, an affine set, and a convex set.

2. $\{x \in \mathbf{R}^n : Ax \leq O\}$ for a given matrix A.

   Solution: This is a cone, hence also convex. But it is not necessarily a subspace or an affine set.

3. $\{x \in \mathbf{R}^n : Ax = b\}$ for a given matrix A and vector b.

   Solution: this set is an affine set, hence also convex. But it is not necessarily a subspace or a cone.

4. $\{x \in \mathbf{R}^n : Ax \leq b\}$ for a given matrix A and vector b.

   Solution for this case: This set is convex. By Proposition 1.13 it suffices to show that it is closed under convex combinations of two elements. So assume $Av^1 \leq b$ and $Av^2 \leq b$, and $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$. We need to verify that $Av \leq b$, where $v = \lambda_1 v^1 + \lambda_2 v^2$. But $Av = A(\lambda_1 v^1 + \lambda_2 v^2) = \lambda_1 Av^1 + \lambda_2 Av_2$. Knowing that $Av^1 \leq b$, $Av^2 \leq b$ and $\lambda_1, \lambda_2 \geq 0$, we deduce that $\lambda_1 Av^1 \leq \lambda_1 b$ and $\lambda_2 Av^2 \leq \lambda_2 b$. Thus $\lambda_1 Av^1 + \lambda_2 Av_2 \leq \lambda_1 b + \lambda_2 b = (\lambda_1 + \lambda_2)b = b$. Therefore $Av \leq b$ and the set is closed under pairwise convex combinations.

$\square$

Proposition 1.13 *A subset $S \subseteq \mathbf{R}^n$ is a subspace (respectively, a cone, an affine set, a convex set) if and only it is closed under all linear (respectively, nonnegative, affine, convex) combinations of pairs of elements.*

PROOF. Exercise. □

Solution for convex combinations: It is immediate that if S is convex then it is closed under convex combinations of pairs of elements. So, conversely, assume that S is closed under convex combinations of pairs of elements. We will prove by induction on $m \geq 1$ that if $v^1, \ldots, v^m \in S$, $\lambda_1, \ldots, \lambda_m \geq 0$, and $\sum_{j=1}^m \lambda_j = 1$, then $v \in S$, where $v = \sum_{j=1}^m \lambda_j v^j$. If $m = 1$ then $\lambda_1 = 1$ and $v \in S$ trivially. If $m = 2$ then $v \in S$ by the assumption that S is closed under convex combinations of pairs of elements. So assume $m > 2$. First consider the case that $\lambda_m = 1$. Then v equals v^m and so $v \in S$. So now assume that $\lambda_m < 1$. Then

$$\begin{aligned} v &= \left(\sum_{j=1}^{m-1} \lambda_j v^j\right) + \lambda_m v^m \\ &= (1-\lambda_m)\left(\sum_{j=1}^{m-1} \frac{\lambda_j}{1-\lambda_m} v^j\right) + \lambda_m v^m \\ &= (1-\lambda_m) w + \lambda_m v^m, \end{aligned}$$

where

$$w = \sum_{j=1}^{m-1} \frac{\lambda_j}{1-\lambda_m} v^j.$$

Note that $1 - \lambda_m > 0$, and so $\frac{\lambda_j}{1-\lambda_m} \geq 0$, $j = 1, \ldots, m-1$. Also note that $\sum_{j=1}^{m-1} \frac{\lambda_j}{1-\lambda_m} = 1$. So $w \in X$ by the induction hypothesis. Hence $v \in S$ since it is convex combination of the pair w and v^m. Therefore S is closed under convex combinations of sets of m elements.

Proposition 1.14 *The intersection of any collection of subspaces (respectively, cones, affine sets, convex sets) is a subspace (respectively, cone, affine set, convex set).*

PROOF. Exercise. □

Solution (for convex sets): Suppose each S_i, $i \in I$, is convex, and $S = \bigcap_{i \in I} S_i$. Let $v^1, \ldots, v^m \in S$ and $\lambda_1, \ldots, \lambda_m \in \mathbf{R}$ such that $\lambda_1 + \cdots + \lambda_m = 1$ and $\lambda_1, \ldots, \lambda_m \geq 0$. Let $v = \sum_{j=1}^m \lambda_i v^i$. Then $v^1, \ldots, v^m \in S_i$ for all $i \in I$. Since S_i is convex, $v \in S_i$ for all i. Hence $v \in \bigcap_{i \in I} S_i = S$, and so S is convex.

Definition 1.15 Let $V \subseteq \mathbf{R}^n$. Define the *linear span* (respectively, *cone*, *affine span*, *convex hull*) of V, denoted $\operatorname{span} V$ (respectively, $\operatorname{cone} V$, $\operatorname{aff} V$, $\operatorname{conv} V$) to be the intersection of all subspaces (respectively, cones, affine sets, convex sets) containing V,

$$\begin{array}{rcl} \operatorname{span} V & = & \bigcap\{S : V \subseteq S,\ S \text{ is a subspace}\}, \\ \operatorname{cone} V & = & \bigcap\{S : V \subseteq S,\ S \text{ is a cone}\}, \\ \operatorname{aff} V & = & \bigcap\{S : V \subseteq S,\ S \text{ is an affine set}\}, \\ \operatorname{conv} V & = & \bigcap\{S : V \subseteq S,\ S \text{ is a convex set}\}. \end{array}$$

Lemma 1.16 *For all $V \subseteq \mathbf{R}^n$, the set $\operatorname{span} V$ (respectively, $\operatorname{cone} V$, $\operatorname{aff} V$, $\operatorname{conv} V$) is a subspace (respectively, cone, affine set, convex set).*

Proof. Exercise. □

Lemma 1.17 *"Linear/nonnegative/affine/convex combinations of linear/nonnegative/affine/convex combinations are linear/nonnegative/affine/convex combinations."*

Proof. Exercise. □

Solution (for convex combinations): Suppose $w^i = \sum_{j=1}^{m_i} \lambda_{ij} v^{ij}$, $i = 1, \ldots, \ell$, where $\lambda_{ij} \geq 0$ for all i, j, and $\sum_{j=1}^{m_i} \lambda_{ij} = 1$ for all i. Let $w = \sum_{i=1}^{\ell} \mu_i w^i$, where $\mu_i \geq 0$ and $\sum_{i=1}^{\ell} \mu_i = 1$. Then $w = \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \mu_i \lambda_{ij} v^{ij}$, where $\mu_i \lambda_{ij} \geq 0$ for all i, j, and

$$\begin{array}{rcl} \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \mu_i \lambda_{ij} & = & \sum_{i=1}^{\ell} \mu_i \sum_{j=1}^{m_i} \lambda_{ij} \\ & = & \sum_{i=1}^{\ell} \mu_i (1) \\ & = & 1. \end{array}$$

Proposition 1.18 *Let $V \subseteq \mathbf{R}^n$. Then $\operatorname{span} V$ (respectively, $\operatorname{cone} V$, $\operatorname{aff} V$, $\operatorname{conv} V$) equals the set of all linear (respectively, nonnegative, affine, convex) combinations of elements of V.*

Proof. Exercise. □

Solution (for convex sets): Let W be the set of all convex combinations of elements of V. By Lemma 1.17, W is closed under convex combinations, and hence is a convex set. Since $V \subseteq W$, we conclude $\operatorname{conv} V \subseteq W$. On the other hand, since $\operatorname{conv} V$ is a convex set containing V, $\operatorname{conv} V$ must contain all convex combinations of elements of V. Hence $W \subseteq \operatorname{conv} V$. Therefore $W = \operatorname{conv} V$.

Lemma 1.19 *Let $v^1, \ldots, v^m \in \mathbf{R}^n$. Let A be the matrix*

$$\begin{bmatrix} v^1 & \cdots & v^m \\ 1 & \cdots & 1 \end{bmatrix}$$

That is to say, A is created by listing the points v^i as columns and then appending a row of 1's. Let $v \in \mathbf{R}^n$. Then v equals the convex combination $\sum_{i=1}^m \lambda_i v^i$ if and only if $\lambda = [\lambda_1, \ldots, \lambda_m]^T$ is a solution of

$$A \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} = \begin{bmatrix} v \\ 1 \end{bmatrix}$$
$$\lambda_1, \ldots, \lambda_m \geq 0$$

PROOF. Exercise. □

Exercise 1.20 What can you say about the rank of the matrix A in the previous problem? □

Theorem 1.21 (Carathéodory) *Suppose x is a convex combination of $v^1, \ldots, v^m \in \mathbf{R}^n$, where $m > n + 1$. Then x is also a convex combination of a subset of $\{v^1, \ldots, v^m\}$ of cardinality at most $n + 1$.*

PROOF. Suggestion: Think about the matrix A in the previous two problems. Assume that the columns associated with positive values of λ_i are linearly dependent. What can you do now? □

Solution: Proof by induction on $m \geq n+1$. The statement is trivially true if $m = n+1$. Assume that $m > n+1$ and that x is a convex combination of $v^1, \ldots, v^m$. Define the matrix A as above and let λ^* be a solution to

$$A\lambda = \begin{bmatrix} x \\ 1 \end{bmatrix}$$
$$\lambda \geq O$$

If any $\lambda_k^* = 0$ then we actually have x written as a convex combination of the elements of $\{v^1, \ldots, v^m\} \setminus \{v^k\}$, a set of cardinality $m - 1$, so the result is true by the induction hypothesis. Thus we assume that $\lambda_j^* > 0$ for all j. The matrix A has more columns than rows ($m > n+1$) so there is a nontrivial element of the nullspace of A, say, μ^*. So $A\mu^* = O$. That μ^* is nontrivial means that at least one μ_j^* is not zero. The last row of A implies that the sum of the μ_j^* equals 0. From this we conclude that at least one μ_j^* is negative. Now consider $\overline{\lambda} = \lambda^* + t\mu^*$, where t is a nonnegative real number. Start with $t = 0$ and increase t until you reach the first value, t^*, for which some component of $\lambda^* + t\mu^*$ becomes zero. In fact,

$$t^* = \min_{j:\mu_j^*<0} \left\{ \frac{\lambda_j^*}{-\mu_j^*} \right\}.$$

Hence $\overline{\lambda}$ is a new solution to

$$\begin{aligned} A\lambda &= \begin{bmatrix} x \\ 1 \end{bmatrix} \\ \lambda &\geq O \end{aligned}$$

with fewer positive entries than λ^*. Let's assume $\overline{\lambda}_k = 0$. Then we have x written as a convex combination of the elements of $\{v^1, \ldots, v^m\} \setminus \{v^k\}$, a set of cardinality $m - 1$, so the result is true by the induction hypothesis.

Definition 1.22 A *V-polytope* is the convex hull of a nonempty finite collection of points in $\mathbf{R}^n$.

Exercise 1.23 Construct some examples of V-polytopes in $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{R}^3$. □

Exercise 1.24 Is the empty set a V-polytope? □

Exercise 1.25 In each of the following cases describe $\operatorname{conv} V$.

1. $V = \{[\pm 1, \pm 1, \pm 1]\} \subset \mathbf{R}^3$.
2. $V = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \subset \mathbf{R}^3$.
3. $V = \{[\pm 1, 0, 0], [0, \pm 1, 0], [0, 0, \pm 1]\} \subset \mathbf{R}^3$.
4. $V = \{0, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{3}{4}, \ldots\} \subset \mathbf{R}$.

□

Theorem 1.26 (Radon) *Let* $V = \{v^1, \ldots, v^m\} \subseteq \mathbf{R}^n$. *If* $m > n + 1$ *then there exists a partition* V_1, V_2 *of* V *such that* $\operatorname{conv} V_1 \cap \operatorname{conv} V_2 \neq \emptyset$. □

Proof. Exercise. □

Solution: Construct the matrix A as in the proof of Theorem 1.21. Because A has more columns than rows ($m > n + 1$) there is a nonzero element μ^* in its nullspace, $A\mu^* = O$. The last row of A then implies that at least one component of μ^* is negative and at least one component is positive. Let $I_\oplus = \{i : \mu_i^* \geq 0\}$ and $I_- = \{i : \mu_i < 0\}$. Note that $I_\oplus$ and I_- are each nonempty. Then

$$\sum_{i \in I_\oplus} \mu_i^* v^i + \sum_{i \in I_-} \mu_i^* v^i = O,$$

$$\sum_{i \in I_\oplus} \mu_i^* + \sum_{i \in I_-} \mu_i^* = 0.$$

So

$$\sum_{i \in I_\oplus} \mu_i^* v^i = \sum_{i \in I_-} (-\mu_i^*) v^i,$$

$$\sum_{i \in I_\oplus} \mu_i^* = \sum_{i \in I_-} (-\mu_i^*).$$

Define $c = \sum_{i \in I_\oplus} \mu_i^* = \sum_{i \in I_-} (-\mu_i^*)$. Note that c is positive. Dividing through by c yields

$$\sum_{i \in I_\oplus} \lambda_i^* v^i = \sum_{i \in I_-} \lambda_i^* v^i,$$

$$\sum_{i \in I_\oplus} \lambda_i^* = \sum_{i \in I_-} \lambda_i^* = 1,$$

$$\lambda \geq O,$$

where $\lambda_i^* = \frac{\mu_i^*}{c}$ if $i \in I_\oplus$, and $\lambda_i^* = \frac{-\mu_i^*}{c}$ if $i \in I_-$. Taking $x = \sum_{i \in I_\oplus} \lambda_i^* v^i = \sum_{i \in I_-} \lambda_i^* v^i$, $V_1 = \{v^i : i \in I_\oplus\}$, and $V_2 = \{v^i : i \in I_-\}$, we see that x is a convex combination of elements of V_1 as well as elements of V_2. So $\operatorname{conv} V_1 \cap \operatorname{conv} V_2 \neq \emptyset$.

Theorem 1.27 (Helly) *Let $\mathcal{V} = \{V_1, \ldots, V_m\}$ be a family of m convex subsets of $\mathbf{R}^n$ with $m \geq n+1$. If every subfamily of $n+1$ sets in $\mathcal{V}$ has a nonempty intersection, then $\cap_{i=1}^m V_i \neq \emptyset$.* $\square$

Proof. Exercise. $\square$

1.2 Linear Inequalities and H-Polytopes

Definition 1.28 Let $a \in \mathbf{R}^n$ and $b_0 \in \mathbf{R}$. Then $a^T x = b_0$ is called a *linear equation*, and $a^T x \leq b_0$ and $a^T x \geq b_0$ are called *linear inequalities.*

Further, if $a \neq O$, then the set $\{x \in \mathbf{R}^n : a^T x = b_0\}$ is called a *hyperplane*, the sets $\{x \in \mathbf{R}^n : a^T x \leq b_0\}$ and $\{x \in \mathbf{R}^n : a^T x \geq b_0\}$ are called *closed halfspaces*, and the sets $\{x \in \mathbf{R}^n : a^T x < b_0\}$ and $\{x \in \mathbf{R}^n : a^T x > b_0\}$ are called *open halfspaces.*

Exercise 1.29 Why do we require $a \neq O$ in the definitions of hyperplanes and halfspaces? $\square$

Definition 1.30 A subset $S \subseteq \mathbf{R}^n$ is *bounded* if there exists a number $M \in \mathbf{R}$ such that $\|x\| \leq M$ for all $x \in S$.

Definition 1.31 We can represent systems of a finite collection of linear equations or linear inequalities in matrix form. For example, the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\leq b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

can be written compactly as

$$Ax \leq b$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

and

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

A nonempty subset of $\mathbf{R}^n$ of the form $\{x \in \mathbf{R}^n : Ax \leq b\}$ is called an *H-polyhedron*. A bounded H-polyhedron is called an *H-polytope*.

Exercise 1.32 Construct some examples of H-polyhedra and H-polytopes in $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{R}^3$. $\square$

Exercise 1.33 Is the empty set an H-polytope? Is $\mathbf{R}^n$ an H-polyhedron? $\square$

Exercise 1.34 Prove that a subset of $\mathbf{R}^n$ described by a finite collection of linear equations and inequalities is an H-polyhedron. $\square$

Exercise 1.35 In each case describe the H-polyhedron $P = \{x : Ax \leq b\}$ where A and b are as given.

1.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

3.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

□

1.3 H-Polytopes are V-Polytopes

Definition 1.36 Suppose $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ is an H-polytope and $\overline{x} \in P$. Partition the linear inequalities into two sets: those that $\overline{x}$ satisfies with equality (the *tight* or *binding* inequalities) and those that $\overline{x}$ satisfies with strict inequality (the *slack* or *nonbinding* inequalities):

$$A^1x \le b^1 \text{ where } A^1\overline{x} = b^1,$$
$$A^2x \le b^2 \text{ where } A^2\overline{x} < b^2.$$

Define $N(\overline{x})$ to be the linear space that is the nullspace of the matrix A^1; i.e., the solution space to $A^1x = O$. Even though this is not an official term in the literature, we will call $N(\overline{x})$ the *nullspace* of $\overline{x}$ (with respect to the system defining P). $\square$

Definition 1.37 Let P be an H-polyhedron and $\overline{x} \in P$ such that $\dim N(\overline{x}) = 0$. Then $\overline{x}$ is called a *vertex* of P. $\square$

Lemma 1.38 *No two different vertices of an H-polyhedron have the same set of tight inequalities.* $\square$

Solution: Assume that $\overline{x}$ and x' are each vertices of the $P = \{x \in \mathbf{R}^n : Ax \le b\}$, and that they have the same associated submatrix A^1 for the set of tight inequalities. Then $A^1\overline{x} = b^1$ and $A^1x' = b^1$, so $A(\overline{x} - x') = O$. Because the nullspace of A^1 is trivial, we conclude $\overline{x} - x' = O$; i.e., $\overline{x} = x'$.

Proposition 1.39 *If P is an H-polyhedron then P has a finite number of vertices.* $\square$

Solution: There is only a finite number of choices for a submatrix A^1.

Lemma 1.40 *Let $P = \{x \in \mathbf{R}^n : Ax \le b\}$ be an H-polytope and $\overline{x} \in P$ such that $\overline{x}$ is not a vertex. Choose any nonzero $\overline{w} \in N(\overline{x})$ and consider the line $\overline{x} + t\overline{w}$. Then there is a positive value of t for which $\overline{x} + t\overline{w}$ is in P and has more tight constraints than $\overline{x}$. Similarly, there is a negative value of t for which $\overline{x} + t\overline{w}$ is in P and has more tight constraints than $\overline{x}$.* $\square$

Theorem 1.41 (Minkowski) *Every H-polytope P is the convex hull of its vertices. Hence every H-polytope is a V-polytope. Suggestion: Prove this by induction on the number of slack inequalities for a point $\overline{x} \in P$.* $\square$

Solution: Let $\overline{x} \in P$ and assume $\overline{x}$ has k slack inequalities.

First assume $k = 0$. Then $A\overline{x} = b$. Claim: $\overline{x}$ is itself a vertex. If not, then there is a nontrivial $\overline{w} \ne O$ in the nullspace of A, and so $A(\overline{x} + t\overline{w}) = b$ for all real t. This implies P contains a line, contradicting that it is a polytope (and hence bounded). Therefore $\overline{x}$ is a vertex and trivially is a convex combination of vertices; namely, itself.

Now assume that $k > 0$. If $\overline{x}$ is itself a vertex, then we are done as before. So assume that $\overline{x}$ is not a vertex. By Lemma 1.40 there exist points $x^1 = \overline{x} + t_1\overline{w}$ with $t_1 > 0$, and $x^2 = \overline{x} + t_2\overline{w}$ with $t < 0$, each in P but having fewer slack inequalities. By the induction hypothesis, each can be written as a convex combination of vertices of P. But also $\overline{x} = \frac{-t_2}{t_1 - t_2}x^1 + \frac{t_1}{t_1 - t_2}x^2$. So $\overline{x}$ is a convex combination of x^1 and x^2. By Lemma 1.17, $\overline{x}$ is itself a convex combination of vertices of P, and we are done.

Exercise 1.42 Determine the vertices of the polytope in $\mathbf{R}^2$ described by the following inequalities:

$$\begin{aligned} x_1 + 2x_2 &\leq 120 \\ x_1 + x_2 &\leq 70 \\ 2x_1 + x_2 &\leq 100 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

□

Exercise 1.43 Determine the vertices of the polytope in $\mathbf{R}^3$ described by the following inequalities:

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_1 + x_3 &\leq 1 \\ x_2 + x_3 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

□

Exercise 1.44 Consider the polytope in $\mathbf{R}^9$ described by the following inequalities:

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 1 \\ x_{21} + x_{22} + x_{23} &= 1 \\ x_{31} + x_{32} + x_{33} &= 1 \\ x_{11} + x_{21} + x_{31} &= 1 \\ x_{12} + x_{22} + x_{32} &= 1 \\ x_{13} + x_{23} + x_{33} &= 1 \\ x_{11} &\geq 0 \\ x_{12} &\geq 0 \\ x_{13} &\geq 0 \\ x_{21} &\geq 0 \\ x_{22} &\geq 0 \\ x_{23} &\geq 0 \\ x_{31} &\geq 0 \\ x_{32} &\geq 0 \\ x_{33} &\geq 0 \end{aligned}$$

Find at least one vertex. □

1.4 V-Polytopes are H-Polytopes

In order to prove the converse of the result in the previous section, i.e., to prove that every V-polytope is an H-polytope, we will need to invoke a procedure called Fourier-Motzkin elimination, which will be discussed later. What we need to know about this procedure for the moment is that whenever we have a polyhedron P described by a system of inequalities in the variables, say, $x_1, \ldots, x_n$, we can eliminate one or more variables of our choosing, say, $x_{k+1}, \ldots, x_n$, to obtain a new system of inequalities describing the projection of P onto the subspace associated with $x_1, \ldots, x_k$.

Theorem 1.45 (Weyl) *If P is a V-polytope, then it is an H-polytope.*

Proof. Assume $P = \operatorname{conv}\{v^1, \ldots, v^m\}$. Consider $P' = \{(r, x) : \sum_{i=1}^m r_i v^i - x = O,\ \sum_{i=1}^m r_i = 1,\ r \geq O\}$. Then $P' = \{(r, x) : \sum_{i=1}^m r_i v^i - x \leq O,\ \sum_{i=1}^m r_i v^i + x \geq O,\ \sum_{i=1}^m r_i \leq 1,\ \sum_{i=1}^m r_i \geq 1,\ r \geq O\}$. Then a description for P in terms of linear inequalities is obtained from that of P' by using Fourier-Motzkin elimination to eliminate the variables $r_1, \ldots, r_m$. Finally, we note that every V-polytope is necessarily a bounded set—we can, for example, bound the norm of any feasible point x in terms of the norms of $v^1, \ldots, v^m$: if $x = \sum_{i=1}^m r_i v^i$ with $\sum_{i=1}^m r_i = 1$ and $r_i \geq 0$ for all $i = 1, \ldots, m$, then

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^m r_i v^i \right\| \\ &\leq \sum_{i=1}^m r_i \|v^i\| \\ &\leq \sum_{i=1}^m \|v^i\|. \end{aligned}$$

□

Exercise 1.46 Experiment with the online demo of "polymake," `http://www.polymake.org/doku.php/boxdoc`, which can convert between descriptions of polytopes as V-polytopes and as H-polytopes. □

2 Theorems of the Alternatives

2.1 Systems of Equations

Let's start with a system of linear equations:

$$Ax = b.$$

Suppose you wish to determine whether this system is feasible or not. One reasonable approach is to use Gaussian elimination. If the system has a solution, you can find a particular one, $\overline{x}$. (You remember how to do this: Use elementary row operations to put the system in row echelon form, select arbitrary values for the independent variables and use back substitution to solve for the dependent variables.) Once you have a feasible $\overline{x}$ (no matter how you found it), it is straightforward to convince someone else that the system is feasible by verifying that $A\overline{x} = b$.

If the system is infeasible, Gaussian elimination will detect this also. For example, consider the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 - x_2 + 3x_3 &= -1 \\ 8x_1 + 2x_2 + 10x_3 + 4x_4 &= 0 \end{aligned}$$

which in matrix form looks like

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right].$$

Perform elementary row operations to arrive at a system in row echelon form:

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array}\right] \left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -8 & 0 & 1 \end{array}\right] \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right] = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right],$$

which implies

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & -2 & 1 \end{array}\right] \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right] = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right].$$

Immediately it is evident that the original system is infeasible, since the resulting equivalent system includes the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = -2$.

This equation comes from multiplying the matrix form of the original system by the third row of the matrix encoding the row operations: $[-4,-2,1]$. This vector satisfies

$$\begin{bmatrix} -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 \\ 8 & 2 & 10 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -2.$$

In matrix form, we have found a vector $\overline{y}$ such that $\overline{y}^T A = O$ and $\overline{y}^T b \neq 0$. Gaussian elimination will always produce such a vector if the original system is infeasible. Once you have such a $\overline{y}$ (regardless of how you found it), it is easy to convince someone else that the system is infeasible.

Of course, if the system is feasible, then such a vector $\overline{y}$ cannot exist, because otherwise there would also be a feasible $\overline{x}$, and we would have

$$0 = O^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) = \overline{y}^T b \neq 0,$$

which is impossible. (Be sure you can justify each equation and inequality in the above chain.) We have established our first Theorem of the Alternatives:

Theorem 2.1 *Either the system*

$$(I) \quad Ax = b$$

has a solution, or the system

$$(II) \quad \begin{aligned} y^T A &= O^T \\ y^T b &\neq 0 \end{aligned}$$

has a solution, but not both.

As a consequence of this theorem, the following question has a "good characterization": Is the system (I) feasible? I will not give an exact definition of this concept, but roughly speaking it means that whether the answer is yes or no, there exists a "short" proof. In this case, if the answer is yes, we can prove it by exhibiting any particular solution to (I). And if the answer is no, we can prove it by exhibiting any particular solution to (II).

Geometrically, this theorem states that precisely one of the alternatives occurs:

1. The vector b is in the column space of A.

2. There is a vector y orthogonal to each column of A (and hence to the entire column space of A) but not orthogonal to b.

2.2 Fourier-Motzkin Elimination — A Starting Example

Now let us suppose we are given a system of linear inequalities

$$Ax \leq b$$

and we wish to determine whether or not the system is feasible. If it is feasible, we want to find a particular feasible vector $\overline{x}$; if it is not feasible, we want hard evidence!

It turns out that there is a kind of analog to Gaussian elimination that works for systems of linear inequalities: Fourier-Motzkin elimination. We will first illustrate this with an example:

$$(I) \quad \begin{array}{c} x_1 - 2x_2 \leq -2 \\ x_1 + x_2 \leq 3 \\ x_1 \leq 2 \\ -2x_1 + x_2 \leq 0 \\ -x_1 \leq -1 \\ 8x_2 \leq 15 \end{array}$$

Our goal is to derive a second system (II) of linear inequalities with the following properties:

1. It has one fewer variable.
2. It is feasible if and only if the original system (I) is feasible.
3. A feasible solution to (I) can be derived from a feasible solution to (II).

(Do you see why Gaussian elimination does the same thing for systems of linear equations?) Here is how it works. Let's eliminate the variable x_1. Partition the inequalities in (I) into three groups, (I_-), (I_+), and (I_0), according as the coefficient of x_1 is negative, positive, or zero, respectively.

$$(I_-) \ \begin{array}{c} -2x_1 + x_2 \leq 0 \\ -x_1 \leq -1 \end{array} \qquad (I_+) \ \begin{array}{c} x_1 - 2x_2 \leq -2 \\ x_1 + x_2 \leq 3 \\ x_1 \leq 2 \end{array} \qquad (I_0) \ 8x_2 \leq 15$$

For each pair of inequalities, one from (I_-) and one from (I_+), multiply by positive numbers and add to eliminate x_1. For example, using the first inequality in each group,

$$\begin{array}{c} (\frac{1}{2})(-2x_1 + x_2 \leq 0) \\ +(1)(x_1 - 2x_2 \leq -2) \\ \hline -\frac{3}{2}x_2 \leq -2 \end{array}$$

System (II) results from doing this for all such pairs, and then also including the inequalities in (I_0):

$$(II)\quad \begin{array}{c} -\frac{3}{2}x_2 \le -2 \\ \frac{3}{2}x_2 \le 3 \\ \frac{1}{2}x_2 \le 2 \\ -2x_2 \le -3 \\ x_2 \le 2 \\ 0x_2 \le 1 \\ 8x_2 \le 15 \end{array}$$

The derivation of (II) from (I) can also be represented in matrix form. Here is the original system:

$$\left[\begin{array}{rr|r} 1 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 8 & 15 \end{array}\right]$$

Obtain the new system by multiplying on the left by the matrix that constructs the desired nonnegative combinations of the original inequalities:

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{rr|r} 1 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 8 & 15 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 15 \end{array}\right].$$

To see why the new system has the desired properties, let's break down this process a bit. First scale each inequality in the first two groups by positive numbers so that each coefficient of x_1 in (I_-) is -1 and each coefficient of x_1 in (I_+) is $+1$.

$$(I_-)\ \begin{array}{c} -x_1 + \frac{1}{2}x_2 \le 0 \\ -x_1 \le -1 \end{array} \quad (I_+)\ \begin{array}{c} x_1 - 2x_2 \le -2 \\ x_1 + x_2 \le 3 \\ x_1 \le 2 \end{array} \quad (I_0)\ 8x_2 \le 15$$

Isolate the variable x_1 in each of the inequalities in the first two groups.

$$(I_-)\ \begin{array}{c} \frac{1}{2}x_2 \le x_1 \\ 1 \le x_1 \end{array} \quad (I_+)\ \begin{array}{c} x_1 \le 2x_2 - 2 \\ x_1 \le -x_2 + 3 \\ x_1 \le 2 \end{array} \quad (I_0)\ 8x_2 \le 15$$

For each pair of inequalities, one from (I_-) and one from (I_+), create a new inequality by "sandwiching" and then eliminating x_1. Keep the inequalities in (I_0).

$$(IIa)\ \begin{array}{ccc} \left\{ \begin{array}{c} \frac{1}{2}x_2 \\ 1 \end{array} \right\} & \le x_1 \le & \left\{ \begin{array}{c} 2x_2 - 2 \\ -x_2 + 3 \\ 2 \end{array} \right\} \\ 8x_2 & \le & 15 \end{array} \longrightarrow (IIb)\ \begin{array}{c} \frac{1}{2}x_2 \le x_1 \le 2x_2 - 2 \\ \frac{1}{2}x_2 \le x_1 \le -x_2 + 3 \\ \frac{1}{2}x_2 \le x_1 \le 2 \\ 1 \le x_1 \le 2x_2 - 2 \\ 1 \le x_1 \le -x_2 + 3 \\ 1 \le x_1 \le 2 \\ 8x_2 \le 15 \end{array}$$

$$\longrightarrow (IIc)\ \begin{array}{c} \frac{1}{2}x_2 \le 2x_2 - 2 \\ \frac{1}{2}x_2 \le -x_2 + 3 \\ \frac{1}{2}x_2 \le 2 \\ 1 \le 2x_2 - 2 \\ 1 \le -x_2 + 3 \\ 1 \le 2 \\ 8x_2 \le 15 \end{array} \longrightarrow (II)\ \begin{array}{c} -\frac{3}{2}x_2 \le -2 \\ \frac{3}{2}x_2 \le 3 \\ \frac{1}{2}x_2 \le 2 \\ -2x_2 \le -3 \\ x_2 \le 2 \\ 0x_2 \le 1 \\ 8x_2 \le 15 \end{array}$$

Observe that the system (II) does not involve the variable x_1. It is also immediate that if (I) is feasible, then (II) is also feasible. For the reverse direction, suppose that (II) is feasible. Set the variables (in this case, x_2) equal to any specific feasible values (in this case we choose a feasible value $\overline{x}_2$). From the way the inequalities in (II) were derived, it is evident that

$$\max \left\{ \begin{array}{c} \frac{1}{2}\overline{x}_2 \\ 1 \end{array} \right\} \le \min \left\{ \begin{array}{c} 2\overline{x}_2 - 2 \\ -\overline{x}_2 + 3 \\ 2 \end{array} \right\}.$$

So there exists a specific value $\overline{x}_1$ of x_1 such that

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \frac{1}{2}\overline{x}_2 \\ 1 \end{array} \right\} & \le \overline{x}_1 \le & \left\{ \begin{array}{c} 2\overline{x}_2 - 2 \\ -\overline{x}_2 + 3 \\ 2 \end{array} \right\} \\
8\overline{x}_2 & \le & 15
\end{array}$$

We will then have a feasible solution to (I).

2.3 Showing our Example is Feasible

From this example, we now see how to eliminate one variable (but at the possible considerable expense of increasing the number of inequalities). If we have a solution to the new system, we can determine a value of the eliminated variable to obtain a solution of the original system. If the new system is infeasible, then so is the original system.

From this we can tackle any system of inequalities: Eliminate all of the variables one by one until a system with no variables remains! Then work backwards to determine feasible values of all of the variables.

In our previous example, we can now eliminate x_2 from system (II):

$$\left[\begin{array}{ccccccc}
2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\
2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\
0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array} \right]
\left[\begin{array}{cc|c}
0 & -3/2 & -2 \\
0 & 3/2 & 3 \\
0 & 1/2 & 2 \\
0 & -2 & -3 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
0 & 8 & 15
\end{array} \right]$$

$$= \left[\begin{array}{cc|c}
0 & 0 & 2/3 \\
0 & 0 & 8/3 \\
0 & 0 & 2/3 \\
0 & 0 & 13/24 \\
0 & 0 & 1/2 \\
0 & 0 & 5/2 \\
0 & 0 & 1/2 \\
0 & 0 & 3/8 \\
0 & 0 & 1
\end{array} \right].$$

Each final inequality, such as $0x_1 + 0x_2 \le 2/3$, is feasible, since the left-hand side is zero and the right-hand side is nonnegative. Therefore the original system is feasible. To find one specific feasible solution, rewrite (II) as

$$\{4/3, 3/2\} \le x_2 \le \{2, 4, 15/8\}\,.$$

We can choose, for example, $\overline{x}_2 = 3/2$. Substituting into (I) (or (IIa)), we require

$$\{3/4, 1\} \le x_1 \le \{1, 3/2, 2\}\,.$$

So we could choose $\overline{x}_1 = 1$, and we have a feasible solution $(1, 3/2)$ to (I).

2.4 An Example of an Infeasible System

Now let's look at the system:

$$(I) \qquad \begin{array}{c} x_1 - 2x_2 \le -2 \\ x_1 + x_2 \le 3 \\ x_1 \le 2 \\ -2x_1 + x_2 \le 0 \\ -x_1 \le -1 \\ 8x_2 \le 11 \end{array}$$

Multiplying by the appropriate nonnegative matrices to successively eliminate x_1 and x_2, we compute:

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{rr|r} 1 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 8 & 11 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 11 \end{array}\right] \qquad (II)$$

and

$$\begin{bmatrix} 2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\ 0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 11 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & 0 & 2/3 \\ 0 & 0 & 8/3 \\ 0 & 0 & 2/3 \\ 0 & 0 & 1/24 \\ 0 & 0 & 1/2 \\ 0 & 0 & 5/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1/8 \\ 0 & 0 & 1 \end{array}\right] \quad (III)$$

Since one inequality is $0x_1+0x_2 \leq -1/8$, the final system (III) is clearly infeasible. Therefore the original system (I) is also infeasible. We can go directly from (I) to (III) by collecting together the two nonnegative multiplier matrices:

$$\begin{bmatrix} 2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\ 0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 2/3 & 0 & 2/3 & 0 & 0 \\ 2/3 & 0 & 2 & 4/3 & 0 & 0 \\ 2/3 & 1 & 0 & 1/3 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 & 0 & 1/8 \\ 1/2 & 2/3 & 0 & 1/3 & 1/2 & 0 \\ 1/2 & 0 & 2 & 1 & 1/2 & 0 \\ 1/2 & 1 & 0 & 0 & 3/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 1/8 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} = M.$$

You can check that $M(I) = (III)$. Since M is a product of nonnegative matrices, it will itself be nonnegative. Since the infeasibility is discovered in the eighth inequality of (III), this comes from the eighth row of M, namely, $[1/2, 0, 0, 0, 1/2, 1/8]$. You can now demonstrate directly to anyone that (I) is infeasible using these nonnegative multipliers:

$$\begin{array}{c} (\frac{1}{2})(x_1 - 2x_2 \leq -2) \\ +(\frac{1}{2})(-x_1 \leq -1) \\ +(\frac{1}{8})(8x_2 \leq 11) \\ \hline 0x_1 + 0x_2 \leq -\frac{1}{8} \end{array}$$

In particular, we have found a nonnegative vector y such that $y^T A = O^T$ but $y^T b < 0$.

2.5 Fourier-Motzkin Elimination in General

Often I find that it is easier to understand a general procedure, proof, or theorem from a few good examples. Let's see if this is the case for you.

We begin with a system of linear inequalities

$$(I) \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m.$$

Let's write this in matrix form as

$$Ax \leq b$$

or

$$A^i x \leq b_i, \quad i = 1, \ldots, m$$

where A^i represents the ith row of A.

Suppose we wish to eliminate the variable x_k. Define

$$\begin{array}{c} I_- = \{i : a_{ik} < 0\} \\ I_+ = \{i : a_{ik} > 0\} \\ I_0 = \{i : a_{ik} = 0\} \end{array}$$

For each $(p,q) \in I_- \times I_+$, construct the inequality

$$-\frac{1}{a_{pk}}(A^p x \leq b_p) + \frac{1}{a_{qk}}(A^q x \leq b_q).$$

By this I mean the inequality

$$\left(-\frac{1}{a_{pk}}A^p + \frac{1}{a_{qk}}A^q\right) x \leq -\frac{1}{a_{pk}}b_p + \frac{1}{a_{qk}}b_q. \tag{1}$$

System (II) consists of all such inequalities, together with the original inequalities indexed by the set I_0.

It is clear that if we have a solution $(\overline{x}_1, \ldots, \overline{x}_n)$ to (I), then $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ satisfies (II). Now suppose we have a solution $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ to (II). Inequality (1) is equivalent to

$$\frac{1}{a_{pk}}(b_p - \sum_{j \neq k} a_{pj}x_j) \leq \frac{1}{a_{qk}}(b_q - \sum_{j \neq k} a_{qj}x_j).$$

As this is satisfied by $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ for all $(p,q) \in I_- \times I_+$, we conclude that

$$\max_{p \in I_-} \left\{ \frac{1}{a_{pk}}(b_p - \sum_{j \neq k} a_{pj}\overline{x}_j) \right\} \leq \min_{q \in I_+} \left\{ \frac{1}{a_{qk}}(b_q - \sum_{j \neq k} a_{qj}\overline{x}_j) \right\}.$$

Choose $\overline{x}_k$ to be any value between these maximum and minimum values (inclusive). Then for all $(p,q) \in I_- \times I_+$,

$$\frac{1}{a_{pk}}(b_p - \sum_{j \neq k} a_{pj}\overline{x}_j) \leq \overline{x}_k \leq \frac{1}{a_{qk}}(b_q - \sum_{j \neq k} a_{qj}\overline{x}_j).$$

Now it is not hard to see that $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_k, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ satisfies all the inequalities in (I). Therefore (I) is feasible if and only if (II) is feasible.

Observe that each inequality in (II) is a nonnegative combination of inequalities in (I), so there is a nonnegative matrix M_k such that (II) is expressible as $M_k(Ax \leq b)$. If we start with a system $Ax \leq b$ and eliminate all variables sequentially via nonnegative matrices $M_1, \ldots, M_n$, then we will arrive at a system of inequalities of the form $0 \leq b'_i$, $i = 1, \ldots, m'$. This system is expressible as $M(Ax \leq b)$, where $M = M_n \cdots M_1$. If no b'_i is negative, then the final system is feasible and we can work backwards to obtain a feasible solution to the original system. If b'_i is negative for some i, then let $\overline{y}^T = M^i$ (the ith row of M), and we have a nonnegative vector $\overline{y}$ such that $\overline{y}^T A = O^T$ and $\overline{y}^T b < 0$.

This establishes a Theorem of the Alternatives for linear inequalities:

Theorem 2.2 *Either the system*

$$(I) \quad Ax \le b$$

has a solution, or the system

$$(II) \quad \begin{array}{c} y^T A = O^T \\ y^T b < 0 \\ y \ge O \end{array}$$

has a solution, but not both.

Note that the "not both" part is the easiest to verify. Otherwise, we would have a feasible $\overline{x}$ and $\overline{y}$ satisfying

$$0 = O^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) \le \overline{y}^T b < 0,$$

which is impossible.

As a consequence of this theorem, we have a good characterization for the question: Is the system (I) feasible? If the answer is yes, we can prove it by exhibiting any particular solution to (I). If the answer is no, we can prove it by exhibiting any particular solution to (II).

2.6 More Alternatives

There are many Theorems of the Alternatives, and we shall encounter more later. Most of the others can be derived from the one of the previous section and each other. For example,

Theorem 2.3 *Either the system*

$$(I) \quad \begin{array}{c} Ax \le b \\ x \ge O \end{array}$$

has a solution, or the system

$$(II) \quad \begin{array}{c} y^T A \ge O^T \\ y^T b < 0 \\ y \ge O \end{array}$$

has a solution, but not both.

Proof. System (I) is feasible if and only if the following system is feasible:

$$(I') \quad \left[\begin{array}{c} A \\ -I \end{array} \right] x \le \left[\begin{array}{c} b \\ O \end{array} \right]$$

System (II) is feasible if and only if the following system is feasible:

$$
(II') \qquad
\begin{array}{c}
\left[\begin{array}{cc} y^T & w^T \end{array}\right] \left[\begin{array}{c} A \\ -I \end{array}\right] = O^T \\
\left[\begin{array}{cc} y^T & w^T \end{array}\right] \left[\begin{array}{c} b \\ O \end{array}\right] < 0 \\
\left[\begin{array}{cc} y^T & w^T \end{array}\right] \geq \left[\begin{array}{cc} O^T & O^T \end{array}\right]
\end{array}
$$

Equivalently,

$$
\begin{array}{c}
y^T A - w^T = O^T \\
y^T b < O \\
y, w \geq O
\end{array}
$$

Now apply Theorem 2.2 to the pair (I'), (II'). □

2.7 Exercises: Systems of Linear Inequalities

Exercise 2.4 Discuss the consequences of having one or more of I_-, I_+, or I_0 being empty during the process of Fourier-Motzkin elimination. Does this create any problems? □

Exercise 2.5 Fourier-Motzkin elimination shows how we can start with a system of linear inequalities with n variables and obtain a system with $n-1$ variables. Explain why the set of all feasible solutions of the second system is a projection of the set of all feasible solutions of the first system. Consider a few examples where $n = 3$ and explain how you can classify the inequalities into types I_-, I_+, and I_0 geometrically (think about eliminating the third coordinate). Explain geometrically where the new inequalities in the second system are coming from. □

Exercise 2.6 Consider a given system of linear constraints. A subset of these constraints is called *irredundant* if it describes the same feasible region as the given system and no constraint can be dropped from this subset without increasing the set of feasible solutions.

Find an example of a system $Ax \leq b$ with three variables such that when x_3, say, is eliminated, the resulting system has a larger irredundant subset than the original system. That is to say, the feasible set of the resulting system requires more inequalities to describe than the feasible set of the original system. Hint: Think geometrically. Can you find such an example where the original system has two variables? □

Exercise 2.7 Use Fourier-Motzkin elimination to graph the set of solutions to the following system:

$$\begin{aligned}
+x_1 + x_2 + x_3 &\le 1 \\
+x_1 + x_2 - x_3 &\le 1 \\
+x_1 - x_2 + x_3 &\le 1 \\
+x_1 - x_2 - x_3 &\le 1 \\
-x_1 + x_2 + x_3 &\le 1 \\
-x_1 + x_2 - x_3 &\le 1 \\
-x_1 - x_2 + x_3 &\le 1 \\
-x_1 - x_2 - x_3 &\le 1
\end{aligned}$$

What is this geometrical object called? □

Exercise 2.8 Prove the following Theorem of the Alternatives: Either the system

$$Ax \ge b$$

has a solution, or the system

$$\begin{aligned}
y^T A &= O^T \\
y^T b &> 0 \\
y &\ge O
\end{aligned}$$

has a solution, but not both. □

Exercise 2.9 Prove the following Theorem of the Alternatives: Either the system

$$\begin{aligned}
Ax &\ge b \\
x &\ge O
\end{aligned}$$

has a solution, or the system

$$\begin{aligned}
y^T A &\le O^T \\
y^T b &> 0 \\
y &\ge O
\end{aligned}$$

has a solution, but not both. □

Exercise 2.10 Prove or disprove: The system

$$(I) \quad Ax = b$$

has a solution if and only if each of the following systems has a solution:

$$(I') \quad Ax \le b \qquad (I'') \quad Ax \ge b$$

□

Exercise 2.11 (The Farkas Lemma). Derive and prove a Theorem of the Alternatives for the following system:

$$\begin{aligned} Ax &= b \\ x &\geq O \end{aligned}$$

Give a geometric interpretation of this theorem when A has two rows. When A has three rows. □

Exercise 2.12 Give geometric interpretations to other Theorems of the Alternatives that we have discussed. □

Exercise 2.13 Derive and prove a Theorem of the Alternatives for the system

$$\begin{gathered} \sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad i \in I_1 \\ \sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i \in I_2 \\ x_j \geq 0, \quad j \in J_1 \\ x_j \text{ unrestricted}, \quad j \in J_2 \end{gathered}$$

where (I_1, I_2) is a partition of $\{1, \ldots, m\}$ and (J_1, J_2) is a partition of $\{1, \ldots, n\}$. □

Exercise 2.14 Derive and prove a Theorem of the Alternatives for the system

$$Ax < b.$$

□

3 Faces of Polytopes

3.1 More on Vertices

For a polytope P, let $\operatorname{vert} P$ denote its set of vertices.

Proposition 3.1 *Suppose $P = \{x \in \mathbf{R}^n : Ax \le b\}$ is a polyhedron and $\overline{x} \in P$. Then $\overline{x}$ is a vertex of P if and only if the set of normal vectors of the binding inequalities for $\overline{x}$ spans $\mathbf{R}^n$ (i.e., $\operatorname{span}\{a^i : {a^i}^T\overline{x} = b_i\} = \mathbf{R}^n$, where a^i denotes row i of A).*

Solution: The normal vectors of the inequalities are precisely the rows of A. Let A^1 be the submatrix corresponding to the binding inequalities for $\overline{x}$. Note that A^1 has n columns. Then $\overline{x}$ is a vertex iff the nullspace of A^1 has dimension 0 iff the columns of A^1 are linearly independent iff the column rank of A^1 equals n iff the row rank of A^1 equals n iff the rows of A span $\mathbf{R}^n$.

Definition 3.2 Suppose $S \subseteq \mathbf{R}^n$, $\overline{x} \in S$, and there exists $c \in \mathbf{R}^n$ such that $\overline{x}$ is the unique maximizer of the linear function $c^T x$ over S. Then $\overline{x}$ is called an *exposed point* of of the set S.

Proposition 3.3 *Every vertex of a polyhedron P is an exposed point.*

Proof. Exercise. □

Solution: Suppose $P = \{x \in \mathbf{R}^n : Ax \le b\}$, and $\overline{x}$ is a vertex of P. Let A^1 be the submatrix of A corresponding to the tight inequalities for $\overline{x}$ and b^1 be the corresponding right-hand sides. Define $c^T = e^T A^1$, where e is a vector of 1's; i.e., c^T is obtained by adding the rows of A^1 together. Then $c^T\overline{x} = (e^T A^1)\overline{x} = e^T(A^1\overline{x}) = e^T b^1$. Now suppose x' is any point in P. Then $c^T x' = (e^T A^1)x' = e^T(A^1 x') \le e^T b^1$ with equality if and only if $A^1 x' = b^1$. But this implies that $x' = \overline{x}$, since $\overline{x}$ is the unique solution to this system.

Lemma 3.4 *Suppose P is a polyhedron, $c \in \mathbf{R}^n$, and $\overline{x} \in P$ maximizes $c^T x$ over P. Assume that*

$$\overline{x} = \sum_{j=1}^{m} \lambda_j v^j,$$

where $v^1, \ldots, v^m$ are vertices of P and $\lambda_1, \ldots, \lambda_m$ are all strictly positive positive numbers summing to 1. Then v^j maximizes $c^T x$ over P, $j = 1, \ldots, m$.

PROOF. Exercise. □

Solution: Let $M = c^T\overline{x}$. Then $c^Tv^j \le M$, $j = 1,\ldots,m$, and so $\lambda_j c^T v^j \le \lambda_j M$. Hence $\sum_{j=1}^m \lambda_j c^T v^j \le \sum_{j=1}^m \lambda_j M = M$ with equality if and only if $c^T v^j = M$ for all $j = 1,\ldots,m$. Hence

$$\begin{aligned} M &= c^T\overline{x} \\ &= \sum_{j=1}^n \lambda_j c^T v^j \\ &\le \sum_{j=1}^n \lambda_j M \\ &= M \end{aligned}$$

forces $c^T v^j = M$, $j = 1,\ldots,m$.

Proposition 3.5 *Every exposed point of a polytope is a vertex.*

PROOF. Exercise. □

Solution: Let $\overline{x}$ be an exposed point and $c^T x$ be a linear function having $\overline{x}$ as its unique maximizer. Write $\overline{x}$ as a convex combination of vertices of P, $\overline{x} = \sum_{j=1}^m \lambda_j v^j$. Discarding v^j for which $\lambda_j = 0$, if necessary, we may assume that $\lambda_j > 0$, $j = 1,\ldots,m$. By Lemma 3.11, $c^T x$ is also maximized at v^j, $j = 1,\ldots,m$. Since $\overline{x}$ is the unique maximizer, we conclude that $v^j = \overline{x}$, $j = 1,\ldots,m$; in particular, $\overline{x}$ is a vertex.

Definition 3.6 Suppose $S \subseteq \mathbf{R}^n$ and $\overline{x} \in S$ such that $\overline{x} \notin \operatorname{conv}(S \setminus \{\overline{x}\})$. Then $\overline{x}$ is called an *extreme point* of the set S.

Proposition 3.7 *Let $\overline{x}$ be a point in a polytope P. Then $\overline{x}$ is a vertex if and only if it is an extreme point.*

PROOF. Exercise. □

Exercise 3.8 Give an example of a convex set S and a point $\overline{x} \in S$ such that $\overline{x}$ is an extreme point but not an exposed point. □

Proposition 3.9 *Let V be a finite set and $P = \operatorname{conv} V$. Then every vertex of P is in V.*

PROOF. Exercise □

Proposition 3.10 *Let V be a finite set, $P = \operatorname{conv} V$, and $v \in V$. Then v is a vertex of P if and only if conv$(V \setminus \{v\}) \neq P$.*

Proof. Exercise $\square$

Proposition 3.11 *Let P be a polytope, $c^T x$ be a linear function, and S be the set of points maximizing $c^T x$ over P. Then S contains at least one vertex of P.*

Proof. Exercise $\square$

3.2 Faces

Definition 3.12 Recall from Exercise 1.11 that every nonempty affine set S is a translate of some linear space L. The *dimension* of S is defined to be the dimension of the linear space L. The empty set is said to have dimension -1. The *dimension*, $\dim T$, of any subset $T \subseteq \mathbf{R}^n$ is then defined to be the dimension of its affine span.

Definition 3.13 Let $P \subset \mathbf{R}^n$ be a polytope of dimension $d > 0$ (a *d-polytope*), and $a^T x$ be a linear function. If the maximizing set F of $a^T x$ over P has dimension j for some $0 \leq j \leq d-1$, then F is a *proper face* of P. If the maximum value of $a^T x$ over P is b, then the hyperplane $\{x : a^T x = b\}$ is a *supporting hyperplane* for F. (Note that $P \subset \{x : a^T x \leq b\}$.)

The empty set and P itself are *improper faces* of P.

If $\dim F = j$ then F is called a *j-face*. Faces of dimension 0, 1, $d-2$, and $d-1$ are called *vertices*, *edges*, *subfacets* or *ridges*, and *facets* of P, respectively.

Proposition 3.14 *If F is a nonempty face of a polytope P, then F is itself a polytope.*

Proof. Exercise. (Just add two inequalities to P.) $\square$

Proposition 3.15 *If F is a proper face of a d-polytope P, then F is a facet if and only if F contains d affinely independent points.*

Proof. Exercise. $\square$

Proposition 3.16 *Let P be a polytope of dimension d in $\mathbf{R}^d$, and F be a proper face of P. Then F is a facet if and only if has a unique supporting hyperplane.*

Proof. Exercise. $\square$

Definition 3.17 Let $Ax \leq b$ be a system of inequalities. Suppose $A_i x \leq b_i$ is the ith inequality in this system (A_i denotes the ith row of A), and $\hat{A}x \leq \hat{b}$ is the system of inequalities resulting from deleting inequality i. If $\{x : Ax \leq b\} = \{x : \hat{A}x \leq \hat{b}\}$, then inequality i is said to be *(geometrically) redundant*; otherwise, it is *(geometrically) irredundant*.

Definition 3.18 Let $Ax \leq b$ be a system of inequalities. Suppose $A_i x \leq b_i$ is the ith inequality in this system, and $\hat{A}x \leq \hat{b}$ is the system of inequalities resulting from deleting inequality i. If there exists $y \geq O$ such that $y^T \hat{A} = A_i$ and $y^T b \leq b_i$, then inequality i is said to be *(algebraically) redundant*; otherwise, it is *(alrebraically) irredundant*.

Proposition 3.19 *Let P be a polytope of dimension d in $\mathbf{R}^d$. Suppose P is defined by the set of inequalities $P = \{x : a^{iT}x \leq b_i,\ i = 1, \ldots, m\}$. Let $H_i = \{x : A_i x = b_i\}$ for some i. Then H_i is the supporting hyperplane to a facet of P if and only if inequality i is geometrically irredundant if and only if inequality i is algebraically irredundant.*

Proof. Exercise. (A Theorem of the Alternatives may be helpful.) □

Proposition 3.20 *Let P be a polytope described by the system $Ax \leq b$, and F be a facet of P. Then one of the inequalities of the system $A_i x \leq b_i$ provides a supporting hyperplane for F. That is to say, F maximizes $A_i x$ over P, and the maximum value is b_i.*

Proof. Exercise. □

Proposition 3.21 *Every nonempty polytope has at least one facet.*

Proof. Exercise. □

Proposition 3.22 *(A face of a face is a face.) Let P be a polytope, F be a proper face of P, and G be a proper face of F (regarding F as a polytope). Then G is a face of P.*

Proof. Exercise. □

Proposition 3.23 *Let P be a polytope and F be a proper face of P. Then F is the convex hull of the vertices of P lying in F.*

Proof. Exercise. □

Proposition 3.24 *Let P be a polytope. Then the set of faces of P, ordered by inclusion, forms a graded poset.*

PROOF. Exercise. (Try induction, knowing that polytopes have facets.) □

Proposition 3.25 *Let F and G be two proper faces of a polytope P. If $F \cap G \neq \emptyset$, then $F \cap G$ is a face of P, and hence is the greatest lower bound of F and G in the face poset.*

PROOF. Exercise. (Add together the linear functions describing F and G.) □

Proposition 3.26 *Let F and G be two proper faces of a polytope P. Then F and G have a least upper bound in the face poset.*

PROOF. Exercise. (Take the intersection of all faces containing both F and G.) □

Theorem 3.27 *The poset of faces of a polytope P is a lattice.*

PROOF. Exercise. □

Definition 3.28 The above lattice is the *face lattice* of P, denoted $\mathcal{F}(P)$.

Proposition 3.29 *Every face of a polytope is the intersection of the facets containing it.*

PROOF. Exercise. □

Definition 3.30 If two polytopes have isomorphic face lattices, then they are said to be *combinatorially equivalent* polytopes.

3.3 Polarity and Duality

Definition 3.31 Let $S \subseteq \mathbf{R}^d$. Then the *polar* of S (with respect to the origin) is the set $S^* = \{x \in \mathbf{R}^d : x^T y \leq 1 \text{ for all } y \in S\}$.

Proposition 3.32 *If $S \subseteq T$, then $S^* \supseteq T^*$.*

PROOF. Exercise. □

Proposition 3.33 *For positive number r, let $B_r \subset \mathbf{R}^d$ be the closed ball of radius r centered at the origin. Then $B_r^* = B_{1/r}$.*

PROOF. Exercise. □

Corollary 3.34 *The set S^* is bounded if and only if S contains B_r for some $r > 0$. The set S^* contains B_r for some $r > 0$ if and only if S is bounded.*

PROOF. Exercise. □

Theorem 3.35 *Let $P \subset \mathbf{R}^d$ be a polytope and $p \in \mathbf{R}^d$. Then either p is in P, or else there is a hyperplane H such that P is in one closed halfspace associated with H and p is in the opposite open halfspace.*

PROOF. Suppose $P = \operatorname{conv} V$ where $V = \{v^1, \ldots, v^n\} \subset \mathbf{R}^d$. We are going to use the Farkas Lemma, Exercise 2.11, which states that either the system

$$\begin{aligned} Ax &= b \\ x &\geq O \end{aligned}$$

has a solution, or else the system

$$\begin{aligned} y^T A &\geq O^T \\ y^T b &< 0 \end{aligned}$$

has a solution, but not both.

Construct the matrix whose columns are the points in V, and then append a row of ones:

$$A = \begin{bmatrix} v^1 & \cdots & v^n \\ 1 & \cdots & 1 \end{bmatrix}.$$

Let b be the vector

$$b = \begin{bmatrix} p \\ 1 \end{bmatrix}.$$

Then $p \in P$ if and only if the system

$$\begin{aligned} Ax &= b \\ x &\geq O \end{aligned}$$

has a solution. The alternative system is:

$$\begin{bmatrix} y^T, y_0 \end{bmatrix} \begin{bmatrix} v^1 & \cdots & v^n \\ 1 & \cdots & 1 \end{bmatrix} \geq O^T$$

$$y^T p + y_0 < 0$$

This system is equivalent to $y^T v^i \geq -y_0$ for all i and $y^T p < -y_0$. Then for any point x in $\operatorname{conv} V$ it is easy to check that $y^T x \leq -y_0$; just write x as a particular convex combination of points in V. Taking H to be the hyperplane $\{x : y^T x = -y_0\}$, this is equivalent to $\operatorname{conv} V$ lying in one of the closed halfspaces associated with H and p lying in the opposite open halfspace. □

Theorem 3.36 *Let $P \subset \mathbf{R}^d$ be a d-polytope containing the origin in its interior. Then $P^{**} = P$.*

PROOF. First, assume $\hat{y} \in P$. For each $x \in P^*$ we have $x^T y \le 1$ for all $y \in P$. Thus $\hat{y}^T x \le 1$ for all $x \in P^*$. Hence $y \in P^{**}$.

Now assume $y \in P^{**}$. Assume that $y \notin P$. Then by Theorem 3.35 there is a hyperplane H such that P lies in one of the closed halfspaces associated with H and y lies in the opposite open halfspace. Knowing that O is in the interior of P, by rescaling the equation of the hyperplane if necessary we may asssume there is a vector c such that all points x in P satisfy $c^T x \le 1$ but $c^T y > 1$. Hence $c \in P^*$ and further $y \notin P^{**}$. This contradiction implies that $y \in P$. $\square$

Proposition 3.37 *Let $P \subset \mathbf{R}^d$ be a d-polytope containing the origin in its interior. Then $P^* = \{x^T y \le 1$ for all $y \in \operatorname{vert} P\}$.*

PROOF. Let $y \in P$. Write y as a convex combination of the vertices of P. Show that the P^* inequality coming from y is an algebraic consequence of the P^* inequalities coming from the vertices of P. $\square$

Corollary 3.38 *Let $P \subset \mathbf{R}^d$ be a d-polytope containing the origin in its interior. Then P^* is a d-polytope.*

PROOF. Exercise. $\square$

Theorem 3.39 *Let $P \subset \mathbf{R}^d$ be a d-polytope containing the origin in its interior. Then the face lattice $\mathcal{F}(P^*)$ is anti-isomorphic to the face lattice $\mathcal{F}(P)$. That is to say, there is an inclusion-reversing bijection between the faces of P^* and the faces of P, matching j-faces F^* of P^* with $(d-j-1)$-faces F of P, $j = -1, \ldots, d$.*

PROOF. Key idea: For each face F of P, consider the points in P^* that also satisfy the equations $\{x : x^T y = 1$ for all $y \in F\}$. This will be the desired face F^* of P^*. You can start with the observation that a point v in P is the convex combination of a set $\{v^1, \ldots, v^n\}$ of vertices of P if and only if the inequality $x^T v \le 1$ is the same convex combination of the inequalities $x^T v^i \le 1$. Then use Propositions 3.23 and 3.29. $\square$

Definition 3.40 P^* is called the *polar dual* polytope to P with respect to O. Any polytope Q combinatorially equivalent to P^* is called a *dual* of P. If $\mathcal{F}(P) \cong \mathcal{F}(P^*)$, then P is said to be *self dual.*

Proposition 3.41 *Any nonempty interval in the face lattice of a polytope P is itself isomorphic to the face lattice of some polytope.*

Proof. Let $[X, Y]$ be an interval in the face lattice of a polytope Q. Call this a *lower interval* if $X = \emptyset$ and an *upper interval* if $Y = Q$. Because Y is itself a polytope, lower intervals are *polytopal* (isomorphic to the face lattice of some polytope; in this case, Y).

Now consider any interval $I = [F, G]$ in $\mathcal{F}(P)$. Then I is an upper interval of the polytopal interval $[\emptyset, G] = \mathcal{F}(G)$. So I is anti-isomorphic to a lower interval of $\mathcal{F}(G^*)$, which is polytopal. Thus I is anti-isomorphic to $\mathcal{F}(H)$ for some polytope H, and hence isomorphic to $\mathcal{F}(H^*)$. $\square$

Definition 3.42 A *simplex* is the convex hull of a set of affinely independent points. A *j-simplex* is a j-dimensional simplex, hence the convex hull of $j + 1$ affinely independent points.

Exercise 3.43 Let $V = \{v^1, \ldots, v^{d+1}\}$ be a set of affinely independent points, and let $P = \operatorname{conv} V$ be a d-simplex. Prove that $\operatorname{conv} S$ is a face of P for every subset of S of P.

Exercise 3.44 Prove that every simplex is self-dual.

Definition 3.45 A d-polytope is *simplicial* if every proper face is a simplex; equivalently, every facet contains exactly d vertices. A d-polytope is *simple* if every vertex is contained in exactly d edges. (Thus, duals of simplicial polytopes are simplicial, and vice versa.)

4 Some Polytopes

4.1 Regular and Semiregular Polytopes

Definition 4.1 A d-dimensional *simplex* or *d-simplex* is the convex hull of $d+1$ affinely independent points.

The *d-cube* is $\operatorname{conv}\{(\pm 1, \pm 1, \ldots, \pm 1)\} \subset \mathbf{R}^d$, where the signs are chosen independently (or any polytope obtained from this one by applying compositions of isometries and scalings).

The *d-cross-polytope* is $\operatorname{conv}\{\pm e^1, \ldots, \pm e^d\} \subset \mathbf{R}^d$, where $e^1, \ldots, e^d$ are the standard unit vectors in $\mathbf{R}^d$ (or any polytope obtained from this one by applying compositions of isometries and scalings).

It turns out that the d-cube and the d-cross-polytope, with the coordinates defined above, are polar to each other.

Definition 4.2 An *isometry* is a composition of reflections through hyperplanes. A *symmetry* of a polytope is an isometry that maps the polytope to itself. Note that the set of symmetries of a given polytope forms a group.

A *full flag* of a d-polytope is a chain of faces $F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{d-1}$, where each F_i is an i-dimensional face (i-face), $i = 0, \ldots, d-1$.

A polytope is *regular* if its group of symmetries is full flag transitive. That is to say, given any two full flags, $F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{d-1}$ and $F'_0 \subset F'_1 \subset F'_2 \subset \cdots \subset F'_{d-1}$, there is a symmetry of the polytope that maps F_i to F'_i, for all i, $i = 0, \ldots, d-1$.

Note that if a polytope is regular, then the definition forces each of its faces to be regular and congruent to each other, and each of its vertex figures to be regular and congruent to each other. (The *vertex figure* is the $(d-1)$-face resulting from "slicing off" a vertex with an "appropriate" hyperplane.)

Exercise 4.3 The simplex $\operatorname{conv}\{e^1, \ldots, e^d\} \subset \mathbf{R}^d$ is a regular $(d-1)$-simplex. The d-cube and the d-cross-polytope are both regular d-polytopes.

The regular convex 2-polytopes are the regular n-gons, $n \geq 3$. These are all self-dual.

Schläfli symbols are a notation to help keep track of the structure of regular polyhedra, starting with ordinary two-dimensional regular polygons. The Schläfli symbol for a polygon with n sides is simply $\{n\}$.

The regular convex 3-polytopes are the five Platonic solids. For a Platonic solid, consisting of regular p-gons with q meeting at each vertex the Schläfli symbol is $\{p, q\}$:

Platonic Solid	Schläfli Symbol
Tetrahedron	$\{3,3\}$
Cube	$\{4,3\}$
Octahedron	$\{3,4\}$
Dodecahedron	$\{5,3\}$
Icosahedron	$\{3,5\}$

Notice that if the Schläfli symbol is $\{p,q\}$, then each face is a regular p-gon, and by truncating any vertex we can get a cross-section that is a regular q-gon. The tetrahedron is self-dual. The cube and the octahedron are duals of each other, as are the dodecahedron and the icosahedron.

There are also four nonconvex regular 3-polytopes, known as the *Kepler-Poinsot solids*:

Kepler-Poinsot Solid	Schläfli Symbol
Small Stellated Dodecahedron	$\{5/2,5\}$
Great Dodecahedron	$\{5,5/2\}$
Great Stellated Dodecahedron	$\{5/2,3\}$
Great Icosahedron	$\{3,5/2\}$

With a generalization of the definition of dual, the first two polytopes above are duals of each other; as are the latter two polytopes.

For regular convex 4-polytopes, each of its 3-faces must be regular 3-polytopes and congruent to each other, and each of its vertex figures must regular 3-polytopes and congruent to each other. It turns out that we can fit three, four, or five regular tetrahedra around a common edge, but there is no room for a sixth. Similarly we can fit three octahedra, three cubes, or three dodecahedra around a common edge, but no more. We cannot fit even three icosahedra around a common edge. These six possibilities can be folded up and extended in four dimensions to create the complete list of the six convex regular four-dimensional polytopes:

Polyhedron	Faces	At Each Edge	At Each Vertex	Schläfli Symbol
5-cell	5 tetrahedra	3 tetrahedra	4 tetrahedra	$\{3,3,3\}$
8-cell	8 cubes	3 cubes	4 cubes	$\{4,3,3\}$
16-cell	16 tetrahedra	4 tetrahedra	8 tetrahedra	$\{3,3,4\}$
24-cell	24 octahedra	3 octahedra	6 octahedra	$\{3,4,3\}$
120-cell	120 dodecahedra	3 dodecahedra	4 dodecahedra	$\{5,3,3\}$
600-cell	600 tetrahedra	5 tetrahedra	20 tetrahedra	$\{3,3,5\}$

The 5-cell is also known as the *regular* 4*-simplex*, the 8-cell as the four-dimensional *hypercube*, and the 16-cell as the four-dimensional *cross polytope*.

Suppose that the Schläfli symbol of a regular polytope is $\{p, q, r\}$. Notice that the first pair of numbers, $\{p, q\}$, describes the 3-face of the polytope. It turns out that the last pair of numbers, $\{q, r\}$, describes what cross-section results when a vertex is truncated, and that this must also be a regular polytope. This severely limits the possibilities for Schläfli symbols, keeping the list to a manageable size.

The 5-cell is self-dual. The 8-cell and the 16-cell are duals of each other. The 24-cell is self-dual. The 120-cell and the 600-cell are duals of each other.

Things get simpler in dimensions beyond 4. For each $d \geq 5$ there are only three regular polytopes: the regular d-simplex, the d-cube, and the d-cross-polytope.

Definition 4.4 A d-polytope is *semiregular* if every $(d-1)$-face is regular (but not all congruent to each other) and its symmetry group is vertex transitive.

The semiregular convex 3-polytopes are the *prisms*, the *antiprisms*, and the 13 *Archimedean solids*.

Blind and Blind [5] describe a complete classification of semiregular polytopes in higher dimensions.

4.2 Polytopes in Combinatorial Optimization

The section oversimplifies the area of combinatorial optimization, but nevertheless describes some important core ideas. A good reference is [29]. The general framework is this: you have a finite set $E = \{e_1, \ldots, e_n\}$, a collection $\mathcal{S}$ of subsets of E, and a function $c : E \to \mathbf{R}$. For any $S \in \mathcal{S}$, define $c(S) = \sum_{e \in S} c(e)$. The goal is to solve the problem

$$\max\{c(S) : S \in \mathcal{S}\}.$$

(Or it could be a minimization problem.)

For each $S \in \mathcal{S}$ define its *characteristic vector* $x(S) \in \mathbf{R}^E$ by

$$x_e(S) = \begin{cases} 1 & \text{if } e \in S, \\ 0 & \text{if } e \notin S. \end{cases}$$

If we now take $c \in \mathbf{R}^E$ to be the vector with coordinates $c_e = c(e)$, our combinatorial optimization problem is equivalent to

$$\max\{c^T x : x = x(S),\ S \in \mathcal{S}\}.$$

Define an associated polytope P to be $\operatorname{conv}\{x(S) : S \in \mathcal{S}\}$. Consider the problem

$$\max\{c^T x : x \in P\}.$$

At first glance, this may seem to be a harder problem, since now we are maximizing over an infinite set. But if we are able to describe P as an H-polytope via a "nice" set of linear inequalities, then we have a *linear programming* problem, for which there are "efficient" algorithms (in terms of the size of the input). Furthermore, linear programming theory assures us that among the optimal points there is a vertex (see Proposition 3.11), and from what we know before (Proposition 3.9), that vertex is in V. So by increasing the feasible set to an infinite set P, we nevertheless arrive at a solution within the original finite set.

Each combinatorial problem is different. Sometimes the associated polytope P is efficiently described by a "small" number of inequalities, and the problem can be solved using general LP algorithms. Sometimes, even if the number of inequalities is not "small", efficient algorithms to solve the problem may be developed, often exploiting knowledge of the combinatorial structure of P. And sometimes the description of the inequalities for P seems intractable.

Example 4.5 The Bipartite Perfect Matching Polytope. Consider the complete bipartite graph $K_{n,n}$, with edge set $E = \{e_{ij} : i = 1, \ldots, n,\ j = 1, \ldots, n\}$. A *perfect matching* is a collection of n edges that share no endpoints. For example, $\{e_{12}, e_{21}, e_{33}\}$ is a perfect matching in $K_{3,3}$, and its characteristic vector is $(x_{11}, x_{12}.x_{13}, x_{21}.x_{22}, x_{23}, x_{31}, x_{32}, x_{33}) = (0, 1, 0, 0, 1, 0, 0, 0, 1)$. The polytope P will have $n!$ vertices, but it turns out that P can be described by $2n$ equations (one of which happens to be redundant) and n^2 inequalities:

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= 1 \text{ for all } i = 1, \ldots, n, \\ \sum_{i=1}^{n} x_{ij} &= 1 \text{ for all } j = 1, \ldots, n, \\ x_{ij} &\geq 0 \text{ for all } i = 1, \ldots, n,\ j = 1, \ldots, n. \end{aligned}$$

There are some very efficient specialized algorithms for solving the maximum weight bipartite perfect matching problem.

Example 4.6 The Matching Polytope. Consider the complete graph K_n, with vertex set $V = \{1, \ldots, n\}$ and edge set E of cardinality $\binom{n}{2}$. A *matching* is a collection of edges (possibly empty) such that no two edges share an endpoint. For vertex v and edge e, define

$$\delta(v, e) = \begin{cases} 1 & \text{if } v \text{ is an endpoint of } e, \\ 0 & \text{if } v \text{ is not an endpoint of } e. \end{cases}$$

One set of valid inequalities for the matching polytope is

$$\begin{aligned} \sum_{e:\delta(v,e)=1} x_e &\leq 1 \text{ for all } v \in V, \\ x_e &\geq 0 \text{ for all } e \in E. \end{aligned}$$

But this does not correctly describe P, even when $n = 3$ (check this!). For subset $T \subseteq V$ and edge $e \in E$, define

$$\bar{\delta}(T, e) = \begin{cases} 1 & \text{if both endpoints of } e \text{ are in } T, \\ 0 & \text{otherwise.} \end{cases}$$

A set of inequalities for the matching polytope is then given by:

$$\begin{aligned} \sum_{e:\delta(v,e)=1} x_e &\leq 1 \text{ for all } v \in V \\ \sum_{e:\bar{\delta}(T,e)=1} x_e &\leq \frac{|T|-1}{2} \text{ for all } T \subseteq V \text{ such that } |T| \geq 3 \text{ and odd,} \\ x_e &\geq 0 \text{ for all } e \in E. \end{aligned}$$

Even though the number of inequalities is exponential in n, there are efficient specialized algorithms exploiting this representation that solve the maximum weight matching problem.

Example 4.7 The Matroid Polytope. A *matroid* is a finite set E and a collection $\mathcal{I}$ of subsets of E (the *independent sets*) that satisfy the following axioms:

Axiom 0. $\emptyset \in \mathcal{I}$.

Axiom 1. If $S \in \mathcal{I}$ and $T \subset S$, then $T \in \mathcal{I}$.

Axiom 2. If $S \in \mathcal{I}$, $T \in \mathcal{I}$, and $|T| > |S|$, then there is some element $e \in T \setminus S$ such that $S \cup \{e\} \in \mathcal{I}$.

One example of a matroid is the *graphic matroid*, in which E is the set of edges of a given graph, and $S \in \mathcal{I}$ if S is a subset of edges containing no cycle. Another is the *linear matroid*, in which E is the index set of the columns of a given matrix A over some field, and $S \in \mathcal{I}$ if S is a subset of columns of A that are linearly independent. It turns out that every graphic matroid is a linear matroid, but there exist matroids that are not linear matroids.

For $S \subseteq E$, define the *rank* of S, $\operatorname{rank} S$, to be $\max\{|T| : T \in \mathcal{I} \text{ and } T \subseteq S\}$. A description of the matroid polytope is given by:

$$\begin{aligned} \sum_{e \in S} x_e &\leq \operatorname{rank} S, \text{ for all } S \subset E, \\ x_e &\geq 0, \text{ for all } e \in E. \end{aligned}$$

The maximum weight independent set in a matroid can be found efficiently using the *Greedy Algorithm*.

Example 4.8 The Dipath Polytope. Given a directed graph G and distinguished vertices $s \neq t$, let E be its set of edges, and $\mathcal{S}$ be subsets of edges corresponding to directed paths from s to t, with no repeated vertices. For vertex v and directed edge e, define

$$\delta^+(v,e) = \begin{cases} 1 & \text{if } e \text{ is directed out of } v, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\delta^-(v,e) = \begin{cases} 1 & \text{if } e \text{ is directed into } v, \\ 0 & \text{otherwise.} \end{cases}$$

Some valid constraints for the dipath polytope are:

$$\begin{aligned} \sum_{e:\delta^+(v,e)=1} x_e - \sum_{e:\delta^-(v,e)=1} &= \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{if } v \neq s,t \end{cases} \\ x_e &\geq 0 \text{ for all } e \in E \end{aligned}$$

But this is not a complete set of constraints—determining the complete set is probably an intractable problem. Nevertheless, there are very efficient algorithms for the minimum weight dipath problem when the function c is nonnegative, such as Dijkstra's Algorithm.

Example 4.9 The Traveling Salesman Polytope. Given a complete directed graph K_n, let E be its set of edges, and $\mathcal{S}$ be subsets of edges corresponding to directed cycles of length n (passing through every vertex). These are directed *Hamilton cycles.* Some valid constraints for the dipath polytope are:

$$\begin{aligned} \sum_{e:\delta^+(v,e)=1} x_e &= 1 \text{ for all vertices } v, \\ \sum_{e:\delta^-(v,e)=1} x_e &= 1 \text{ for all vertices } v, \\ x_e &\geq 0 \text{ for all } e \in E. \end{aligned}$$

Again, this is not a complete set of constraints and determining the complete set is probably an intractable problem. There are no known efficient algorithms for finding a maximum or minimum weight directed Hamilton cycle.

5 Representing Polytopes

5.1 Schlegel Diagrams

A *Schlegel diagram* provides a convenient way to represent a p-polytope in $d-1$ dimensions, most often useful when $d = 4$ or $d = 3$. Let P be a d-polytope, and F be a facet of P. Choose a point $p \notin P$ "sufficiently close" to a point in the relative interior of F. Project each face G of P using central projection onto F in the direction of p. The resulting decomposition of F into polyhedra is a Schlegel diagram of P with respect to F and p.

5.2 Gale Transforms and Diagrams

Let $V = \{v^1, \ldots, v^n\}$ be a finite subset of $\mathbf{R}^d$ and $P = \operatorname{conv} V$. Assume that $\dim P = d$. Let A be the $(d+1) \times n$ matrix

$$A = \begin{bmatrix} v^1 & \cdots & v^n \\ 1 & \cdots & 1 \end{bmatrix}.$$

Exercise 5.1 Prove that $\operatorname{rank} A = d + 1$.

Now find a basis for the nullspace of A, and list these $n-d-1$ basis elements as the rows of an $(n-d-1) \times n$ matrix $\overline{A}$. Label the columns of this matrix $\overline{v}^1, \ldots, \overline{v}^n$. Then $\{\overline{v}^1, \ldots, \overline{v}^n\} \subset \mathbf{R}^{n-d-1}$ is a *Gale transform* of V.

Exercise 5.2 Prove that $\overline{v}^1 + \cdots + \overline{v}^n = O$.

Theorem 5.3 *Let $S \subset \{1, \ldots, n\}$. Then the set $\{v^i : i \in S\}$ is $V \cap F$ for some face F of P if and only if the set* $\operatorname{conv} \{\overline{v}^i : i \notin S\}$ *contains O in its relative interior; equivalently, if and only if O is a positive linear combination of the points in $\{\overline{v}^i : i \notin S\}$.*

Proof. Use the criterion that S is the set of vertices of a face if and only if there is a hyperplane H such that S lies in H and all of the other vertices in V lie in one of the open halfspaces associated with H. □

A *Gale diagram* of P is a set $\overline{\overline{V}} = \{\overline{\overline{v}}^1, \ldots, \overline{\overline{v}}^n\} \subset \mathbf{R}^{n-d-1}$ that is combinatorially equivalent to $\overline{V}$, in the sense that for every subset $T \subset \{1, \ldots, n\}$, O is in the relative interior of $\operatorname{conv} \{\overline{v}^i : i \in T\}$ if and only if O is in the relative interior of $\operatorname{conv} \{\overline{\overline{v}}^i : i \in T\}$. For example, one way to get a Gale diagram from a Gale transform is to independently scale each of the points by positive amounts.

Exercise 5.4 How can you tell from a Gale diagram whether or not every point in V is a vertex of P?

Exercise 5.5 Assuming V is the vertex set of P, how can you tell from a Gale diagram whether or not P is simplicial?

Exercise 5.6 Suppose V has exactly $d+2$ points. What does a Gale transform tell you about Radon partitions (see Theorem 1.26? What if V has more than $d+2$ points?

Exercise 5.7 Determine a formula for the number of different combinatorial types of (unlabeled) d-polytopes with exactly $d+2$ vertices.

Exercise 5.8 Determine a formula for the number of different combinatorial types of (unlabeled) simplicial d-polytopes with exactly $d+3$ vertices.

Remark 5.9 Given any set $\overline{V} = \{\overline{v}^1, \ldots, \overline{v}^n\} \subset \mathbf{R}^{n-d-1}$ such that O is in the interior of the convex hull of $\overline{V}$, it may be regarded as the Gale diagram of some d-polytope P. First find positive numbers $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1\overline{v}^1 + \cdots \lambda_n\overline{v}^n = O$. Construct the matrix

$$\overline{\overline{A}} = \left[\begin{array}{ccc} \lambda_1\overline{v}^1 & \cdots & \lambda_n\overline{v}^n \end{array} \right].$$

Then the rank of $\overline{\overline{A}}$ equals $n-d-1$ and you can find a $d \times n$ matrix A of the form

$$A = \left[\begin{array}{ccc} v^1 & \cdots & v^n \\ 1 & \cdots & 1 \end{array} \right]$$

such that the rows of A form a basis for the nullspace of $\overline{\overline{A}}$. Then define $V = \{v^1, \ldots, v^n\}$ and $P = \operatorname{conv} V$.

6 Euler's Relation

Recall that $\mathcal{F}(P)$ denotes the set of all faces of a polytope P, both proper and improper. Let $\mathcal{F}(\text{bd}\, P) := \mathcal{F}(P)\backslash\{P\}$. We denote the number of j-dimensional faces (*j-faces*) of P by $f_j(P)$ (or simply f_j when the polyhedron is clear) and call $f(P) := (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ the *f-vector* of P. The empty set is the unique face of dimension -1 and P is the unique face of dimension d, so $f_{-1}(P) = 1$ and $f_d(P) = 1$.

A very big problem is to understand/describe $f(\mathcal{P}^d) := \{f(P) : P \text{ is a } d\text{-polytope}\}$!

6.1 Euler's Relation for 3-Polytopes

By now you have all probably encountered the formula "$V - E + F = 2$" for convex three-dimensional polytopes.

Theorem 6.1 (Euler's Relation) *If P is a 3-polytope, then $f_0 - f_1 + f_2 = 2$.*

For historical notes, see [3], [13]. [16], and [22]. Before Euler stated his formula, Descartes discovered a theorem from which Euler's formula could be deduced, but it does not appear that Descartes explicitly did so.

Proof by "immersion". Position the polytope P in a container so that no two vertices are at the same vertical level (have the same z-coordinate). Fill the container with water. Count the contribution of a face to the expression $f_0 - f_1 + f_2$ at the moment when it is submerged. At the very beginning, only the bottom vertex becomes submerged, so at this point $f_0 - f_1 + f_2 = 1 - 0 + 0 = 1$. When a later vertex v (but not the top one) is submerged, that vertex now contributes, as do the downward edges incident to v (let's say there are k of them) and the $k-1$ facets between these k-edges. So the contribution of these newly submerged faces to $f_0 - f_1 + f_2$ is $1 - k + (k-1) = 0$. Thus $f_0 - f_1 + f_2$ remains equal to 1. But when the top vertex v is submerged, all of its incident edges (let's say there are k of them) are submerged, as well as k incident facets. The contribution of these newly submerged faces to $f_0 - f_1 + f_2$ is $1 - k + k = 1$, so at the end $f_0 - f_1 + f_2 = 2$. $\square$

Exercise 6.2 What happens when you apply this proof technique to a "polyhedral torus"? $\square$

Proof by "projection and destruction". Choose a facet F of P. Find a point q outside of the polytope P but "sufficiently close" to a point in the relative interior of F. Make a Schlegel diagram of P by projecting the vertices and the edges of P onto F using central projection towards q. Now you have a connected, planar graph G in the plane. There is a bijection between the regions of H determined by G, and the facets of P. Let f_0, f_1, and f_2 be the number of vertices, edges, and regions, respectively, of G. Find a cycle in G, if there is one (a sequence of vertices and edges $v_0e_1v_1e_2v_2\cdots v_ke_k$ where $k>2$, the v_i are distinct, and e_i joins v_{i-1} to v_i, $i=1,\ldots,k$). Delete one edge of the cycle. Then f_1 and f_2 each drop by one (why?). So $f_0-f_1+f_2$ does not change. Repeat this step until no cycles remain. Now find a vertex incident to precisely one of the remaining edges (why does the nonexistence of cycles imply that such a vertex exists?). Delete this vertex and this edge. Then f_0 and f_1 each drop by one. So $f_0-f_1+f_2$ does not change. Repeat this step until the graph is reduced to a single vertex and a single region with no edges (this is where the connectivity of G comes into play). At this stage $f_0-f_1+f_2=1-0+1=2$, so it must have been equal to two at the start as well. $\square$

This proof applies to arbitrary connected planar graphs.

Proof by "shelling". Build up the boundary of the polytope facet by facet, keeping track of $f_0-f_1+f_2$ as you go. Be sure each new facet F (except the last) meets the union of the previous ones in a single path of vertices and edges along the boundary of F. Suppose the first facet has k edges. Then at this point $f_0-f_1+f_2=k-k+1=1$. Suppose a later facet F (but not the last) has k edges but meets the previous facets along a path with ℓ edges and $\ell+1$ vertices, $\ell<k$. Then F as a whole increases $f_0-f_1+f_2$ by $k-k+1=1$, but we must subtract off the contribution by the vertices and edges in the intersection, which is $(\ell+1)-\ell=1$. So there is no net change to $f_0-f_1+f_2$. The very last facet increases only f_2 (by one), giving the final result $f_0-f_1+f_2=2$. $\square$

At this point, however, it is not obvious that every 3-polytope can be built up in such a way, so this proof requires more work to make it secure.

Proof by algebra. Let $\mathcal{F}_j$ denote the set of all j-faces of P, $j=-1,0,1,2$. For $j=-1,0,1,2$ define vector spaces $X_j=\mathbf{Z}_2^{\mathcal{F}_j}$ over $\mathbf{Z}_2$ with coordinates indexed by the j-faces. If you like, you may think of a bijection between the vectors of X_j and the subsets of $\mathcal{F}_j$. In particular, $\dim X_j=f_j$. For $j=0,1,2$ we are going to define a linear *boundary* map $\partial_j: X_j\to X_{j-1}$. Assume $x=(x_F)_{F\in\mathcal{F}_j}$. Let $\partial_j(x)=(y_G)_{G\in\mathcal{F}_{j-1}}$ be defined by

$$y_G=\sum_{F:G\subset F}x_F.$$

Define also $\partial_{-1}: X_{-1}\to 0$ by $\partial_{-1}(x)=0$, and $\partial_3: 0\to X_2$ by $\partial_3(0)=0$. You should be able to verify that $\partial_{j-1}\partial_j(x)$ equals zero for all $x\in X_j$, $j=0,1,2$ (why?). Set $B_j=\partial_{j+1}(X_{j+1})$ and $C_j=\ker\partial_j$, $j=-1,0,1,2$. By the previous observation, $B_j\subseteq C_j$. The subspaces B_j are called *j-boundaries* and the subspaces C_j are called *j-cycles*. Note that $\dim X_j = \dim B_{j-1} + \dim C_j$, $j = 0, 1, 2$. Finally define the quotient spaces $H_j = C_j/B_j$, $j = -1, \ldots, 2$. (These are the *(reduced) homology spaces* of the boundary complex of P over $\mathbf{Z}_2$.) Then $\dim H_j = \dim C_j - \dim B_j$. It turns out that B_j actually equals C_j, $j = -1, 0, 1$ (prove this!), so for these values of j we have $\dim B_j = \dim C_j$ and $\dim H_j = 0$. Observe that $\dim B_2 = 0$ and $\dim C_2 = 1$ (why?). So $\dim H_2 = 1$, and we have

$$\begin{aligned}
1 &= \dim H_2 - \dim H_1 + \dim H_0 - \dim H_{-1} \\
&= (\dim C_2 - \dim B_2) - (\dim C_1 - \dim B_1) + (\dim C_0 - \dim B_0) - (\dim C_{-1} - \dim B_{-1}) \\
&= -\dim B_2 + (\dim C_2 + \dim B_1) - (\dim C_1 + \dim B_0) + (\dim C_0 + \dim B_{-1}) - \dim C_{-1} \\
&= 0 + \dim X_2 - \dim X_1 + \dim X_0 - 1 \\
&= f_2 - f_1 + f_0 - 1.
\end{aligned}$$

This implies $2 = f_2 - f_1 + f_0$. □

Exercise 6.3 If P is a 3-polytope, prove that $\partial_{j-1}\partial_j(x) = 0$ equals zero for all $x \in X_j$, $j = 0, 1, 2$. □

Exercise 6.4 Begin thinking about which of the above proofs might generalize to higher dimensions, and how. □

6.2 Some Consequences of Euler's Relation for 3-Polytopes

Exercise 6.5 For a 3-polytope P, let p_i denote the number of faces that have i vertices (and hence i edges), $i = 3, 4, 5, \ldots$. (The vector $(p_3, p_4, p_5, \ldots)$ is called the *p-vector* of P.) Let q_i denote the number of vertices at which i faces (and hence i edges) meet, $i = 3, 4, 5, \ldots$.

1. Prove

   (a) $3p_3 + 4p_4 + 5p_5 + 6p_6 + \cdots = 2f_1$.

   (b) $2f_1 \geq 3f_2$.

   (c) $3q_3 + 4q_4 + 5q_5 + 6q_6 + \cdots = 2f_1$.

   (d) $2f_1 \geq 3f_0$.

   (e) $f_2 \leq 2f_0 - 4$.

   (f) $f_2 \geq \frac{1}{2}f_0 + 2$.

2. Label the horizontal axis in a coordinate system f_0 and the vertical axis f_2. Graph the region for which the above two inequalities (1e) and (1f) hold.

3. Consider all integral points (f_0, f_2) lying in the above region. Can you find a formula for the number of different possible values of integral values f_2 for a given integral value of f_0?

4. Prove that no 3-polytope has exactly 7 edges.

5. Think of ways to construct 3-polytopes that achieve each possible integral point (f_0, f_2) in the region.

6. Prove that $f_0 - f_1 + f_2 = 2$ is the unique linear equation (up to nonzero multiple) satisfied by the set of f-vectors of all 3-polytopes.

7. **Characterization of $f(\mathcal{P}^3)$.** Describe necessary and sufficient conditions for (f_0, f_1, f_2) to be the f-vector of a 3-polytope.

□

Exercise 6.6

1. Prove the following inequalities for 3-polytopes.

   (a) $6 \le 3f_0 - f_1$.

   (b) $6 \le 3f_2 - f_1$.

   (c) $12 \le 3p_3 + 2p_4 + 1p_5 + 0p_6 - 1p_7 - 2p_8 - \cdots$.

2. Prove that every 3-polytope must have at least one face that is a triangle, quadrilateral, or pentagon.

3. Prove that every 3-polytope must have at least one vertex at which exactly 3, 4, or 5 edges meet.

4. A truncated icosahedron (soccer ball) is an example of a 3-polytope such that (1) each face is a pentagon or a hexagon, and (2) exactly three faces meet at each vertex. Prove that any 3-polytope with these two properties must have exactly 12 pentagons.

□

Exercise 6.7 Suppose P is a 3-polytope with the property that each facet has exactly n edges and exactly m edges meet at each vertex. (The *Platonic* (or *regular*) *solids* satisfy these criteria.) List all the possible pairs (m, n). □

Exercise 6.8 Suppose P is a 3-polytope with the property that exactly a_k k-gons meet at each vertex, $k = 3, \ldots, \ell$. (The *semiregular solids*, including the *Archimedean solids*, satisfy this criterion.) Determine f_0, f_1, and f_2 in terms of $a_3, \ldots, a_\ell$. $\Box$

Exercise 6.9 Recall from plane geometry that for any polygon, the sum of the exterior angles (the amount by which the interior angle falls short of π) always equals 2π. There is a similar formula for 3-polytopes. For each vertex calculate by how much the sum of the interior angles of the polygons meeting there falls short of 2π. Then sum these shortfalls over all the vertices. Prove that this sum equals 4π. $\Box$

6.3 Euler's Relation in Higher Dimensions

Grünbaum [16] credits Schläfli [40] for the discovery of Euler's Relation for d-polytopes in 1852 (though published in 1902). He explains that there were many other discoveries of the relation in the 1880's, but these relied upon the unproven assumption that the boundary complexes of polytopes were suitably "shellable." The first real proof seems to be by Poincaré [35, 36] in 1899 during the time when the Euler characteristic of manifolds was under development. Perhaps the first completely elementary proof without algebraic overtones is that of Grünbaum [16]. The proof that we give below is a bit different, but still a sibling of Grünbaum's proof.

Theorem 6.10 (Euler-Poincaré Relation) *If P is a d-polytope, then*

$$\chi(P) := \sum_{j=0}^{d-1} (-1)^j f_j(P) = 1 - (-1)^d.$$

The subset $\{(f_0, \ldots, f_{d-1}) \in \mathbf{R}^d : \sum_{j=0}^{d-1} (-1)^j f_j = 1 - (-1)^d\}$ is sometimes called the *Euler hyperplane*.

Two alternative expressions of this result are

$$\hat{\chi}(P) := \sum_{j=-1}^{d-1} (-1)^{d-j-1} f_j(P) = 1,$$

and

$$\sum_{j=-1}^{d} (-1)^j f_j(P) = 0.$$

Proof. Assume that P is a subset of $\mathbf{R}^d$. Choose a vector $c \in \mathbf{R}^d$ such that $c^T v$ is different for each vertex v of P (why can this be done?). Order the vertices of P, $v_1, \ldots, v_n$, by increasing value of $c^T v_i$. For $k = 1, \ldots, n$, define $S_k(P) := \{F \subset P : F \text{ is a face of } P \text{ such that } c^T x \le c^T v_k \text{ for all } x \in F\}$. (Clearly $S_n(P) = \mathcal{F}(\operatorname{bd} P)$, the set of all faces of P.) We will prove that

$$\hat{\chi}(S_k(P)) = \begin{cases} 0, & k = 1, \ldots, n-1, \\ 1, & k = n. \end{cases}$$

Our proof is by double induction on d and n. It is easy to check its validity for $d = 0$ and $d = 1$, so fix $d \ge 2$. When $k = 1$, $S_1(P)$ consists of the empty set and v_1, so $\hat{\chi}(S_1(P)) = 0$. Assume $k \ge 2$. Then

$$\begin{aligned} \hat{\chi}(S_k(P)) &= \hat{\chi}(S_{k-1}(P)) + \hat{\chi}(S_k(P) \setminus S_{k-1}(P)) \\ &= \hat{\chi}(S_k(P) \setminus S_{k-1}(P)). \end{aligned}$$

Let Q be a *vertex figure* of P at v_k. This is constructed by choosing a hyperplane H for which v_k and the set $\{v_1, \ldots, v_n\} \setminus v_k$ are in opposite open halfspaces associated with H. Then define $Q := P \cap H$. Let $m := f_0(Q)$. It is a fact that Q is a $(d-1)$-polytope, and there is a bijection between the j-faces F of P containing v_k and the $(j-1)$-faces $F \cap H$ of Q. Moreover, the faces in the set $S_k(P) \setminus S_{k-1}(P)$ correspond to the faces in $S_\ell(Q)$, defined using the same vector c, for some $\ell \le m$, with $\ell < m$ if and only if $k < n$. (You may need to perturb H slightly to ensure that $c^T x$ is different for each vertex of Q.) Therefore

$$\begin{aligned} \hat{\chi}(S_k(P) \setminus S_{k-1}(P)) &= \sum_{j=-1}^{d-1} (-1)^{d-j-1} f_j(S_k(P) \setminus S_{k-1}(P)) \\ &= \sum_{j=0}^{d-1} (-1)^{d-j-1} f_j(S_k(P) \setminus S_{k-1}(P)) \\ &= \sum_{j=0}^{d-1} (-1)^{d-j-1} f_{j-1}(S_\ell(Q)) \\ &= \sum_{j=-1}^{d-2} (-1)^{d-j-2} f_j(S_\ell(Q)) \\ &= \hat{\chi}(S_\ell(Q)) \\ &= \begin{cases} 0, & \ell < m, \\ 1, & \ell = m. \end{cases} \end{aligned}$$

□

If we are looking for linear equations satisfied by members of $f(\mathcal{P}^d)$, we are done:

Theorem 6.11 *Up to scalar multiple, the relation $\chi(P) = 1 - (-1)^d$ is the unique linear equation satisfied by all $(f_0, \ldots, f_{d-1}) \in f(\mathcal{P}^d)$, $d \geq 1$.*

Proof. We prove this by induction on d. For $d = 1$, the relation states $f_0 = 2$, and the result is clear. Assume $d \geq 2$. Suppose $\sum_{j=0}^{d-1} a_j f_j = b$ is satisfied by all $f \in f(\mathcal{P}^d)$, where not all a_j are zero. Let Q be any $(d-1)$-polytope and suppose $f(Q) = (\hat{f}_0, \ldots, \hat{f}_{d-2})$. Let P_1 be a *pyramid* over Q and P_2 be a *bipyramid* over Q. Such polytopes are created by first realizing Q as a subset of $\mathbf{R}^d$. The pyramid P_1 is constructed by taking the convex hull of Q and any particular point not in the affine span of Q. The bipyramid P_2 is constructed by taking the convex hull of Q and any particular line segment L such that the intersection of Q and L is a point in the relative interiors of both Q and L. It is a fact that

$$\begin{array}{rcl} f(P_1) & = & (\hat{f}_0 + 1, \hat{f}_1 + \hat{f}_0, \hat{f}_2 + \hat{f}_1, \ldots, \hat{f}_{d-2} + \hat{f}_{d-3}, 1 + \hat{f}_{d-2}), \\ f(P_2) & = & (\hat{f}_0 + 2, \hat{f}_1 + 2\hat{f}_0, \hat{f}_2 + 2\hat{f}_1, \ldots, \hat{f}_{d-2} + 2\hat{f}_{d-3}, 2\hat{f}_{d-2}). \end{array}$$

Both P_1 and P_2 are d-polytopes, so

$$\sum_{j=0}^{d-1} a_j f_j(P_1) = b,$$

$$\sum_{j=0}^{d-1} a_j f_j(P_2) = b.$$

Subtracting the first equation from the second yields

$$a_0 + a_1\hat{f}_0 + a_2\hat{f}_1 + a_3\hat{f}_2 + \cdots + a_{d-2}\hat{f}_{d-3} + a_{d-1}(\hat{f}_{d-2} - 1) = 0$$

and so

$$a_1\hat{f}_0 + a_2\hat{f}_1 + a_3\hat{f}_2 + \cdots + a_{d-2}\hat{f}_{d-3} + a_{d-1}\hat{f}_{d-2} = a_{d-1} - a_0$$

for all $\hat{f} \in f(\mathcal{P}^{d-1})$. This relation cannot be the trivial relation; otherwise $a_1 = \cdots = a_{d-1} = 0$ and $a_{d-1} - a_0 = 0$, which forces $a_j = 0$ for all j. So by induction this relation must be a nonzero scalar multiple of

$$\hat{f}_0 - \hat{f}_1 + \hat{f}_2 - \cdots + (-1)^{d-2}\hat{f}_{d-2} = 1 - (-1)^{d-1}.$$

Thus $a_1 \neq 0$, $a_j = (-1)^{j-1}a_1$, $j = 1, \ldots, d-1$, and $a_{d-1} - a_0 = (1 - (-1)^{d-1})a_1$, so

$$\begin{array}{rcl} a_0 & = & a_{d-1} - (1 - (-1)^{d-1})a_1 \\ & = & (-1)^{d-2}a_1 - a_1 + (-1)^{d-1}a_1 \\ & = & -a_1. \end{array}$$

From this we see that $a_j = (-1)^j a_0$, $j = 0, \ldots, d-1$, which in turn forces $b = (1 - (-1)^d)a_0$. Therefore $\sum_{j=1}^{d-1} a_j f_j = b$ is a nonzero scalar multiple of Euler's Relation. □

6.4 Gram's Theorem

We now turn to an interesting geometric relative of Euler's Relation. Gram's Theorem is described in terms of solid angle measurement in [16]; in which the history of the theorem and its relatives is discussed (Gram's contribution is for $d = 3$). The form we give here, and its consequence for volume computation, is summarized from Lawrence [23]. See also [24, 25].

Suppose P is a d-polytope in $\mathbf{R}^d$. Each facet F_i has a unique supporting hyperplane H_i. Let H_i^+ be the closed halfspace associated with H_i containing P.

For every face F, whether proper or not, define

$$K_F := \bigcap_{i:F\subseteq H_i} H_i^+.$$

Note in particular that $K_\emptyset = P$ and $K_P = \mathbf{R}^d$. Define the function $a_F : \mathbf{R}^d \to \mathbf{R}$ by

$$a_F(x) = \begin{cases} 1, & x \in K_F, \\ 0, & x \notin K_F. \end{cases}$$

Theorem 6.12 *If P is a d-polytope, then*

$$\sum_{F:-1\leq \dim F\leq d} (-1)^{\dim F} a_F(x) = 0 \textit{ for all } x \in \mathbf{R}^d.$$

Equivalently,

$$\sum_{F:0\leq \dim F\leq d} (-1)^{\dim F} a_F(x) = \begin{cases} 1, & x \in P, \\ 0, & x \notin P. \end{cases}$$

The proof that the above sum equals one when $x \in P$ follows easily from Euler's Relation. The case $x \notin P$ is more easily understood after we have discussed shellability. □

7 The Dehn-Sommerville Equations

7.1 3-Polytopes

If P is a 3-polytope, then of course $f_0 - f_1 + f_2 = 2$. But if every facet of P is a triangle, then we can say more: $3f_2 = 2f_1$. These two equations are linearly independent, and every equation satisfied by f-vectors of all such 3-polytopes is a consequence of these two.

Exercise 7.1 Prove that the set of f-vectors of 3-polytopes, all of whose facets are triangles, is $\{(f_0, 3f_0 - 6, 2f_0 - 4) : f_0 \in \mathbf{Z},\ f_0 \geq 4\}$. $\square$

What is the situation in higher dimensions?

A d-polytope P is called *simplicial* if every proper j-face of P is a j-simplex. Equivalently, it is enough to know that every facet of P is a $(d-1)$-simplex.

Simplicial polytopes are dual to simple polytopes, and $(a_0, \ldots, a_{d-1})$ is the f-vector of some simplicial d-polytope if and only if $(a_{d-1}, \ldots, a_0)$ is the f-vector of some simple d-polytope (why?). Our goal in this section is to learn more about $f(\mathcal{P}_s^d)$, the set of f-vectors of the collection $\mathcal{P}_s^d$ of all simplicial polytopes, but it turns out to be easier to view the situation from the simple standpoint first.

7.2 Simple Polytopes

Let v be a vertex of a simple d-polytope Q. Let E be the collection of the d edges of Q containing v. It is a fact that there is a bijection between subsets S of E of cardinality j and j-faces of Q containing v; namely, that unique face of Q containing v and S, but not $E \setminus S$. (Can you see why this won't be true in general if Q is not simple?)

Now assume that Q is a simple d-polytope in $\mathbf{R}^d$, and choose a vector $c \in \mathbf{R}^d$ such that $c^T v$ is different for every vertex v of Q. As in Section 6.3, order the vertices $v_1, \ldots, v_n$ of Q according to increasing value of $c^T x$, and define the sets $S_k := S_k(Q)$. It is a fact that for every nonempty face F of Q there is a unique point of F that maximizes $c^T x$ over all $x \in F$, and that this point is one of the vertices of Q—the unique vertex v_k such that $F \in S_k \setminus S_{k-1}$. Orient each edge uv of Q in the direction of increasing value of $c^T x$; i.e., so that it is pointing from vertex u to vertex v if $c^T u < c^T v$.

Choose a vertex v_k, and assume that there are exactly i edges pointing into v_k (so v_k has *indegree* i and *outdegree* $d - i$). By the above observations, the number of j-faces of $S_k \setminus S_{k-1}$ equals $\binom{i}{j}$. Let h_i^c be the number of vertices of Q with indegree i. Then since each j-face of Q appears exactly once in some $S_k \setminus S_{k-1}$ (necessarily for some vertex v_k of indegree at least

i), we see that

$$f_j = \sum_{i=j}^{d} \binom{i}{j} h_i^c, \ j = 0, \ldots, d. \tag{2}$$

Exercise 7.2 Define the polynomials

$$\hat{f}(Q, t) = \sum_{j=0}^{d} f_j t^j$$

and

$$\hat{h}(Q, t) = \sum_{i=0}^{d} h_i^c t^i.$$

1. Prove $\hat{f}(Q, t) = \hat{h}(Q, t + 1)$.
2. Prove $\hat{h}(Q, t) = \hat{f}(Q, t - 1)$.
3. Conclude

$$h_i^c = \sum_{j=i}^{d} (-1)^{i+j} \binom{j}{i} f_j, \ i = 0, \ldots, d. \tag{3}$$

□

The above exercise proves the surprising fact that the numbers h_i^c are independent of the choice of c. In particular, $h_i^{-c} = h_i^c$ for all $i = 0, \ldots, d$. But the vertices of indegree i with respect to $-c$ are precisely the vertices of outdegree $d - i$ with respect to $-c$, hence the vertices of indegree $d - i$ with respect to c. Therefore, $h_i^{-c} = h_{d-i}^c$ for all i. Dispensing with the now superfluous superscript c, we have

$$h_i = h_{d-i}, \ i = 0, \ldots, d, \tag{4}$$

for every simple d-polytope Q. These are the *Dehn-Sommerville Equations* for simple polytopes. We may, if we wish, drop the superscript in equation (3), and use this formula to *define* h_i, $i = 0, \ldots, d$, for simple d-polytopes. The vector $h := (h_0, \ldots, h_d)$ is the *h-vector* of the simple polytope Q.

Exercise 7.3

1. Calculate the h-vector of a 3-cube.
2. Calculate the f-vector and the h-vector for a d-cube with vertices $(\pm 1, \ldots, \pm 1)$. (The *d-cube* is the Cartesian product of the line segment $[-1, 1]$ with itself d-times.) Suggestion: Use induction on d and the fact that every facet of a d-cube is a $(d - 1)$-cube.

□

7.3 Simplicial Polytopes

We now return to the simplicial viewpoint. For a simplicial d-polytope P, let Q be a simple d-polytope dual to P. For $i = 0, \ldots, d$,

$$\begin{aligned} h_i &= h_i(Q) \\ &= h_{d-i}(Q) \\ &= \sum_{k=d-i}^{d} (-1)^{d-i+k} \binom{k}{d-i} f_k(Q) \\ &= \sum_{k=d-i}^{d} (-1)^{d-i+k} \binom{k}{d-i} f_{d-k-1}(P). \end{aligned}$$

Let $j = d - k$. Then

$$h_i = \sum_{j=0}^{i} (-1)^{i+j} \binom{d-j}{d-i} f_{j-1}(P), \; i = 0, \ldots, d. \tag{5}$$

We take equation (5) as the *definition* of $h_i(P) := h_i$, $i = 0, \ldots, d$, and let $h(P) := (h_0(P), \ldots, h_d(P))$ be the *h-vector* of the simplicial polytope P. The following two theorems follow immediately.

Theorem 7.4 (Dehn-Sommerville Equations) *If P is a simplicial d-polytope, then $h_i(P) = h_{d-i}(P)$, $i = 0, \ldots, \lfloor (d-1)/2 \rfloor$.*

Theorem 7.5 *If P is a simplicial d-polytope, then $h_i \geq 0$, $i = 0, \ldots, d$.*

In Theorem 7.4, $\lfloor x \rfloor$ is the *greatest integer function*, defined to be $\lfloor x \rfloor := \max\{y : y \leq x$ and y is an integer$\}$.

For a simplicial polytope P, define the polynomials

$$f(P, t) = \sum_{j=0}^{d} f_{j-1} t^j$$

and

$$h(P, t) = \sum_{i=0}^{d} h_i t^i.$$

Exercise 7.6

1. Prove $h(P,t) = (1-t)^d f(P, \frac{t}{1-t})$.

2. Prove $f(P,t) = (1+t)^d h(P, \frac{t}{1+t})$.

3. Prove

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i, \ j = 0, \ldots, d. \tag{6}$$

$\square$

Exercise 7.7

1. Find the formulas for h_0, h_1, and h_d in terms of the f_j.

2. Find the formulas for f_{-1}, f_0, and f_{d-1} in terms of the h_i.

3. Prove that $h_0 = h_d$ is equivalent to Euler's Relation for simplicial d-polytopes.

$\square$

Exercise 7.8 Characterize $h(\mathcal{P}_s^3)$; i.e., characterize which vectors (h_0, h_1, h_2, h_3) are h-vectors of simplicial 3-polytopes. $\square$

Exercise 7.9 Show that the number of monomials of degree s in at most r variables is $\binom{r+s-1}{s}$. $\square$

Exercise 7.10 Show that the number of monomials of degree s in exactly r variables (i.e., each variable appears with positive power) is $\binom{s-1}{r-1}$. $\square$

Exercise 7.11 Use Exercise 7.9 to show that the coefficient of t^s in the expansion of $\frac{1}{(1-t)^r} = (1+t+t^2+\cdots)^r$ is $\binom{r+s-1}{s}$. $\square$

Exercise 7.12

Prove that $f(P, \frac{t}{1-t})$ formally expands to the series $\sum_{\ell=0}^{\infty} H_\ell(P) t^\ell$ where

$$H_\ell(P) = \begin{cases} 1, & \ell = 0, \\ \sum_{j=0}^{\ell-1} f_j(P) \binom{\ell-1}{j}, & \ell > 0, \end{cases}$$

(taking $f_j(P) = 0$ if $j \geq d$). $\square$

Exercise 7.13 Prove Stanley's observation that the f-vector can be derived from the h-vector by constructing a triangle in a manner similar to Pascal's triangle, but replacing the right-hand side of the triangle by the h-vector. The f-vector emerges at the bottom. Consider the example of the octahedron.

$$\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & 1 & & \mathbf{3} & & & \\
 & & 1 & & 4 & & \mathbf{3} & & \\
 & 1 & & 5 & & 7 & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{6} & & \mathbf{12} & & \mathbf{8} & &
\end{array}$$

By subtracting instead of adding, one can convert the f-vector to the h-vector in a similar way. □

Exercise 7.14 What are the f-vector and the h-vector of a d-simplex? □

Exercise 7.15 Let P be a simplicial convex d-polytope and let Q be a simplicial convex d-polytope obtained by building a shallow pyramid over a single facet of P. Of course, this increases the number of vertices by one. Show that the h-vector of Q is obtained by increasing $h_i(P)$ by one, $i = 1, \ldots, d-1$. □

Exercise 7.16 A simplicial convex d-polytope is called *stacked* if it can be obtained from a d-simplex by repeatedly building shallow pyramids over facets. What do the h-vector and the f-vector of a stacked d-polytope with n vertices look like? □

Exercise 7.17 Let P be a d-polytope with n vertices such that $f_{j-1}(P) = \binom{n}{j}$, $j = 0, \ldots, \lfloor d/2 \rfloor$. Prove that $h_i(P) = \binom{n-d+i-1}{i}$, $i = 0, \ldots, \lfloor d/2 \rfloor$. Suggestion: Consider the lower powers of t in $f(P,t)$ and $h(P,t)$. □

Exercise 7.18

1. Suppose P is a simplicial d-polytope and P' is a bipyramid over P. What is the relationship between $h(P)$ and $h(P')$?

2. Let P_1 be any 1-polytope (line segment), and let P_k be a bipyramid over P_{k-1}, $k = 2, 3, \ldots$. (Such P_k are combinatorially equivalent to *d-cross-polytopes*, which are dual to d-cubes.) Find formulas for $h(P_k)$ and $f(P_k)$.

□

7.4 The Affine Span of $f(\mathcal{P}_s^d)$

For a simplicial d-polytope P, $d \geq 1$, consider the equation $h_i = h_{d-i}$. Obviously if $i = d-i$ then the equation is trivial, so let's assume that $0 \leq i \leq \lfloor (d-1)/2 \rfloor$ (in particular, $d-i > i$). Then, as a linear combination of $f_{-1}, \ldots, f_{d-1}$, h_{d-i} contains the term f_{d-i-1}, whereas h_i does not. So the equation is nontrivial, $i = 0, \ldots, \lfloor (d-1)/2 \rfloor$. Clearly these equations form a linearly independent set of $\lfloor (d-1)/2 \rfloor + 1 = \lfloor (d+1)/2 \rfloor$ linear equations, so the dimension of the affine span of f-vectors $(f_0, \ldots, f_{d-1})$ of simplicial d-polytopes is at most $d - \lfloor (d+1)/2 \rfloor = \lfloor d/2 \rfloor$.

Let $m = \lfloor d/2 \rfloor$. To verify that the $\dim \operatorname{aff} f(\mathcal{P}_s^d) = m$, we need to find a collection of $m+1$ affinely independent f-vectors. (The notation aff denotes *affine span*.) Fortunately, there is a class of simplicial d-polytopes, called *cyclic polytopes*, which accomplishes this. We'll study cyclic polytopes a bit later, but for now it suffices to know that $f_{j-1} = \binom{n}{j}$, $j = 0, \ldots, m$, for cyclic d-polytopes $C(n,d)$ with n vertices.

Exercise 7.19 Prove that the set $\{f(C(n,d)) : n = d+1, \ldots, d+m+1\}$ is affinely independent. Suggestion: Write these vectors as rows of a matrix, throw away all but the first m columns, append an initial column of 1's, and then show that this matrix has full row rank by subtracting adjacent rows from each other. □

Theorem 7.20 *The dimension of* $\operatorname{aff} f(\mathcal{P}_s^d)$ *is* $\lfloor d/2 \rfloor$, *and* $\operatorname{aff} f(\mathcal{P}_s^d) = \{(f_0, \ldots, f_{d-1}) : h_i = h_{d-i},\ i = 0, \ldots, \lfloor (d-1)/2 \rfloor\}$.

The Dehn-Sommerville Equations can be expressed directly in terms of the f-vector. Here is one way (see [16]):

Theorem 7.21 *If* $f \in f(\mathcal{P}_s^d)$ *then*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j = (-1)^{d-1} f_k,\ -1 \leq k \leq d-1.$$

The dual result for simple polytopes (see [6]) is:

Theorem 7.22 *If* $f = (f_0, \ldots, f_d)$ *is the* f*-vector of a simple* d*-polytope, then*

$$\sum_{j=0}^{i} (-1)^j \binom{d-j}{d-i} f_j = f_i,\ i = 0, \ldots, d.$$

7.5 Vertex Figures

Let's return to a simplicial d-polytope P. Assume that v is a vertex of P, and Q is a vertex figure of P at v. Define B to be the collection of faces of P that do not contain v. It is a fact that Q is a simplicial $(d-1)$-polytope. In the following formulas we take $f_{-2}(Q) = h_{-1}(Q) = h_d(Q) = 0$.

Theorem 7.23 *Let P, Q, and B be as above. Then*

1. $f_j(P) = f_j(B) + f_{j-1}(Q)$, $j = -1, \ldots, d-1$.
2. $h_i(P) = h_i(B) + h_{i-1}(Q)$, $i = 0, \ldots, d$.
3. $h_i(Q) - h_{i-1}(Q) = h_i(B) - h_{d-i}(B)$, $i = 0, \ldots, d$.

Proof.

1. This is clear because every j-face of P either does not contain v, in which case it is a j-face of B, or else does contain v, in which case it corresponds to a $(j-1)$-face of Q.
2. Expressing (1) in terms of polynomials, we get
$$f(P,t) = f(B,t) + tf(Q,t).$$
So
$$(1-t)^d f(P, \frac{t}{1-t}) = (1-t)^d f(B, \frac{1}{1-t}) + (1-t)^d \frac{t}{1-t} f(Q, \frac{t}{1-t}),$$
$$h(P,t) = h(B,t) + th(Q,t),$$
and equating coefficients of t^i gives (2).
3. The Dehn-Sommerville Equations for P and Q are equivalent to the statements
$$h(P,t) = t^d h(P, \frac{1}{t})$$
and
$$h(Q,t) = t^{d-1} h(Q, \frac{1}{t}).$$
Therefore
$$\begin{aligned} h(B,t) - t^d h(B, \tfrac{1}{t}) &= h(P,t) - th(Q,t) - t^d h(P, \tfrac{1}{t}) + t^d \tfrac{1}{t} h(Q, \tfrac{1}{t}) \\ &= t^{d-1} h(Q, \tfrac{1}{t}) - th(Q,t) \\ &= h(Q,t) - th(Q,t). \end{aligned}$$
Equating coefficients of t^i gives (3). □

This theorem tells us that the h-vectors, and hence the f-vectors, of both P and Q, are completely determined by the h-vector, and hence the f-vector, of B. We can use (3) to iteratively compute $h_0(Q), h_1(Q), h_2(Q), \ldots$, and then determine $h(P)$ from (2).

If we think of the boundary complex of P as a hollow $(d-1)$-dimensional "simplicial sphere", then B is a $(d-1)$-dimensional "simplicial ball", and the faces on the "boundary" of B correspond to the faces of Q. Actually (though I haven't defined the terms), the Dehn-Sommerville Equations apply to any simplicial sphere, so this theorem can be generalized to prove that the f-vector of the boundary of any simplicial ball is completely determined by the f-vector of the ball itself.

Example 7.24 Suppose P is a simplicial 7-polytope, v is a vertex of P, and B is defined as above. Assume that

$$f(B) = (11, 55, 165, 314, 365, 234, 63).$$

Let's find $f(P)$ and $f(Q)$.

$$f(B,t) = 1 + 11t + 55t^2 + 165t^3 + 314t^4 + 365t^5 + 234t^6 + 63t^7,$$

$$\begin{aligned} h(B,t) &= (1-t)^7 f(B, \frac{t}{1-t}) \\ &= 1 + 4t + 10t^2 + 20t^3 + 19t^4 + 7t^5 + 2t^6. \end{aligned}$$

Set up an "addition" for $h(B)$, $h(Q)$, and $h(P)$:

$$\begin{array}{c|cccccccc} h(B) & 1 & 4 & 10 & 20 & 19 & 7 & 2 & 0 \\ h(Q) & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline h(P) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

The missing two rows must each be symmetric, which forces the solution

$$\begin{array}{c|cccccccc} h(B) & 1 & 4 & 10 & 20 & 19 & 7 & 2 & 0 \\ h(Q) & + & 1 & 3 & 6 & 7 & 6 & 3 & 1 \\ \hline h(P) & 1 & 5 & 13 & 26 & 26 & 13 & 5 & 1 \end{array}$$

So

$$\begin{aligned} h(Q,t) &= 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6, \\ h(P,t) &= 1 + 5t + 13t^2 + 26t^3 + 26t^4 + 13t^5 + 5t^6 + t^7. \end{aligned}$$

Using $f(Q,t) = (1+t)^6 h(\frac{t}{1+t})$ and $f(P,t) = (1+t)^7 h(\frac{t}{1+t})$, we compute

$$\begin{aligned} f(Q) &= (9, 36, 81, 108, 81, 27), \\ f(P) &= (12, 64, 201, 395, 473, 315, 90). \end{aligned}$$

□

Exercise 7.25 Let B be as in the previous theorem and let $\operatorname{int} B$ be the set of faces of B that do not correspond to faces of Q (are not on the "boundary" of B). Prove that $h(\operatorname{int} B) = (h_d(B), \ldots, h_0(B))$. $\square$

Messing around a bit, we can come up with an explicit formula for $f(Q)$ in terms of $f(B)$ (we omit the proof):

Theorem 7.26 *Assume that B and Q are as in the previous theorem. Then*

$$f_k(Q) = f_k(B) + (-1)^d \sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(B), \quad -1 \le k \le d-2.$$

Comparing this to Theorem 7.21, we see that $f(Q)$ measures the deviation of $f(B)$ from satisfying the Dehn-Sommerville Equations.

It is a fact that every unbounded simple d-polyhedron R is dual to a certain B as occurs above, and that the collection of unbounded faces of R is dual to Q. So the previous theorem can be used to get an explicit formula for the number of unbounded faces of R in terms of the f-vector of R.

Theorem 7.27 *If $f = (f_0, \ldots, f_d)$ is the f-vector of a simple d-polyhedron R, and if f_j^u is the number of unbounded j-faces of R, $j = 1, \ldots, d$, then*

$$f_i^u(R) = f_i - \sum_{j=0}^{i} (-1)^j \binom{d-j}{d-i} f_j, \;\; i = 0, \ldots, d.$$

7.6 Notes

In 1905 Dehn [14] worked on the equations for $d = 4$ and $d = 5$ and conjectured the existence of analogous equations for $d > 5$. Sommerville [42] derived the complete system of equations for arbitrary d in 1927. Klee [20] in 1964 rediscovered the Dehn-Sommerville Equations in the more general setting of manifolds and incidence systems. In addition to d-polytopes, the equations hold also for simplicial $(d-1)$-spheres, triangulations of homology $(d-1)$-spheres, and Klee's Eulerian $(d-1)$-spheres. See [16] for more historical details and generalizations. McMullen and Walkup [33] (see also [32]) introduced the important notion of the h-vector (though they used the letter g). (I have heard, however, that Sommerville may have also formulated the Dehn-Sommerville Equations in a form equivalent to $h_i = h_{d-i}$—I have yet to check this.) Stanley [43, 44, 45, 46, 47, 48] made the crucial connections between the h-vector and algebra, some of which we shall discuss later.

For more on the Dehn-Sommerville Equations, see [6, 16, 32, 51].

8 Shellability

We mentioned that early "proofs" of Euler's Relation assumed that (the boundaries of) polytopes are shellable. But this wasn't established until 1970 by Bruggesser and Mani [8]—with a wonderful insight reaffirming that hindsight is 20/20.

Recall for any d-polytope P that $\mathcal{F}(P)$ denotes the set of all faces of P, both proper and improper, and $\mathcal{F}(\mathrm{bd}\, P) = \mathcal{F}(P) \setminus \{P\}$. Suppose S is a nonempty subset of $\mathcal{F}(\mathrm{bd}\, P)$ with the property that if F and G are two faces of P with $F \subseteq G \in S$, then $F \in S$; that is to say, the collection S is *closed under inclusion*.

We define S to be *shellable* if the following conditions hold:

1. For every j-face F in S there is a facet of P in S containing F (S is *pure*).

2. The facets in S can be ordered $F_1, \ldots, F_n$ such that for every $k = 2, \ldots, n$, $T_k := \mathcal{F}(F_k) \cap (\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1}))$ is a shellable collection of faces of the $(d-1)$-polytope F_k.

Such an ordering of the facets in S is called a *shelling order*. We say that a d-polytope P is *shellable* if $\mathcal{F}(\mathrm{bd}\, P)$ is shellable.

We note first that if S consists of a single facet F of P and all of the faces contained in F, then condition (1) is trivially true and condition (2) is vacuously true. So every 0-polytope is shellable (there is only one facet—the empty set). It is easy to check that every 1-polytope is shellable (try it). Condition (2) implies in particular that the intersection $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ is nonempty, since the empty set is a member of $\mathcal{F}(F_1), \ldots, \mathcal{F}(F_k)$.

Exercise 8.1

1. Let P be a 2-polytope. Use the definition to characterize when a set S of faces of P is shellable. □

2. Investigate the analogous question when P is a 3-polytope.

Exercise 8.2 Let P be a d-simplex. Then P has $d+1$ vertices, and every subset of vertices determines a face of P. Let $\{F_1, \ldots, F_m\}$ be any subset of facets of P. Prove that $S := \mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_m)$ is shellable, and the facets of S can be shelled in any order. □

Theorem 8.3 (Bruggesser-Mani 1971) *Let P be a d-polytope. Then P is shellable. Further, there is a shelling order $F_1, \ldots, F_n$ of the facets of P such that for every $k = 1, \ldots, n$, there is a shelling order $G_1^k, \ldots, G_{n_k}^k$ of the facets of F_k for which $\mathcal{F}(F_k) \cap (\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1}))$ equals $\mathcal{F}(G_1^k) \cup \cdots \cup \mathcal{F}(G_\ell^k)$ for some $0 \le \ell \le n_k$. Moreover, $\ell = n_k$ if and only if $k = n$.*

The case $\ell = 0$ occurs if and only if $k = 1$, and is just a sneaky way of saying that $\mathcal{F}(\mathrm{bd}\, F_1)$ is shellable.

Imagine that P is a planet and that you are in a rocket resting on one of the facets of P. Take off from the planet in a straight line, and create a list of the facets of P in the order in which they become "visible" to you (initially one one facet is visible). Proceed "to infinity and beyond," returning toward the planet along the same line but from the opposite direction. Continue adding facets to your list, but this time in the order in which they disappear from view. The last facet on the list is the one you land on. Bruggesser and Mani proved that this is a shelling order (though I believe they traveled by balloon instead of by rocket). The proof given here is a dual proof.

Proof. We'll prove this by induction on d. It's easy to see the result is true if $d = 0$ and $d = 1$, so assume $d > 1$. Let $P^* \subset \mathbf{R}^d$ be a polytope dual to P. Choose a vector $c \in \mathbf{R}^d$ such that $c^T v$ is different for each vertex v of P^*. Order the vertices $v_1, \ldots, v_n$ by increasing value of $c^T v_i$. The vertices of P^* correspond to the facets $F_1, \ldots, F_n$ of P. We claim that that this is a shelling order.

For each $i = 1, \ldots, n$, define $\mathcal{F}^*(v_i)$ to be the set of faces of P^* that contain v_i (including P^* itself). Let $S_k(P^*) = \mathcal{F}^*(v_1) \cup \cdots \cup \mathcal{F}^*(v_k)$. We will prove that $S_k(P^*)$ is dual (anti-isomorphic) to a shellable collection of faces of P, $k = 1, \ldots, n$.

The result follows from the following observations about the duality between P and P^*:

1. As mentioned above, the facets $F_1, \ldots, F_n$ of P correspond to the vertices $v_1, \ldots, v_n$ of P^*.

2. For each k, $\mathcal{F}(F_k)$ is dual to the set $\mathcal{F}^*(v_k)$.

3. For each k, the facets $G_1^k, \ldots, G_{n_k}^k$ of F_k correspond to the edges of P^* that contain v_k, which in turn correspond to the vertices $v_1^k, \ldots, v_{n_k}^k$ of a vertex figure F_k^* of P^* at v_k. The facets $G_1^k, \ldots, G_{n_k}^k$ are to be ordered by the induced ordering of $v_1^k, \ldots, v_{n_k}^k$ by c. (In constructing the vertex figure, be sure that its vertices have different values of $c^T v_i^k$.)

4. For each k and i, the set $\mathcal{F}(G_i^k)$ is dual to the set $\mathcal{F}^*(v_i^k)$, defined to be the set of faces of F_k^* that contain v_i^k.

5. For each k, $\mathcal{F}^*(v_k) \cap (\mathcal{F}^*(v_1) \cup \cdots \cup \mathcal{F}^*(v_{k-1}))$ is dual to the collection of faces in T_k. It consists of all of the faces of P^* containing both v_k and some "lower" vertex v_i, $i = 1, \ldots, k-1$. Equivalently, these are the faces of P^* containing at least one edge joining v_k to some lower vertex v_i, $i = 1, \ldots, k-1$. Thus, looking at the vertex figure

F_k^*, this set of faces corresponds to the set $S_\ell(F_k^*) = \mathcal{F}^*(v_1^k) \cup \cdots \cup \mathcal{F}^*(v_\ell^k)$ for some $\ell \le n_k$. This set $S_\ell(F_k^*)$ is dual to a shellable collection of faces of F_k by induction. Further, $\ell = n_k$ if and only if $k = n$. $\square$

Exercise 8.4 (Line Shellings) Assume $P \subset \mathbf{R}^d$ is a d-polytope containing the origin O in its interior. Let $F_1, \ldots, F_n$ be the facets of P and let $H_1, \ldots, H_n$ be the respective supporting hyperplanes for these facets. Choose a direction $c \in \mathbf{R}^d$ and define the line $L := \{tc : t \in \mathbf{R}\}$. Assume that c has been chosen such that as you move along L you intersect the various H_i one at a time (why does such a line exist?). By relabeling, if necessary, assume that as you start from O and move in one direction along L (t positive and increasing) you encounter the H_i in the order $H_1, \ldots, H_\ell$. Now move toward O from infinity along the other half of L (t negative and increasing) and assume that you encounter the remaining H_i in the order $H_{\ell+1}, \ldots, H_n$. Prove that $F_1, \ldots, F_n$ constitutes a shelling order by examining the polar dual P^*. (Such shellings are called *line shellings*.) $\square$

Exercise 8.5 Find a 2-polytope P and some ordering of the facets of P that is a shelling, but not a line shelling, regardless of the location of O. $\square$

Exercise 8.6 Let P be a d-polytope and $F_1, \ldots, F_n$ be a line shelling order of its facets. For $k = 1, \ldots, n$, let $S_k = \mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_k)$. For any subset of faces S of P define

$$\hat{\chi}(S) := \sum_{j=-1}^{d-1} (-1)^{d-j-1} f_j(S).$$

Prove Euler's Relation by showing that

$$\hat{\chi}(S_k) = \begin{cases} 0, & k = 1, \ldots, n-1, \\ 1, & k = n. \end{cases}$$

$\square$

Exercise 8.7 Let P be a d-polytope. If $F_1, \ldots, F_n$ is a line-shelling of P, then the only time $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ contains *all* of the facets of F_k is when $k = n$. Show that the same is true for arbitrary shelling orders, not just line shellings. Suggestion: Use Exercise 8.6. $\square$

Exercise 8.8 Let P be a d-polytope and v be any vertex of P. Prove that there is a shelling order of P such that the set of facets containing v are shelled first. $\square$

Exercise 8.9 Let P be a simplicial d-polytope. Explain how the h-vector of P can be calculated from a shelling order of its facets. Do this in the following way: Assume that $F_1, \ldots, F_n$ is a shelling order of the facets of a simplicial d-polytope. Prove that for every $k = 1, \ldots, n$ there is a face G_k in $\mathcal{F}(F_k)$ such that $\mathcal{F}(F_k) \cap (\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1}))$ is the set of all faces of F_k not containing G_k. Then show that

$$h_i(S_k) = \begin{cases} h_i(S_{k-1}) + 1, & i = f_0(G_k), \\ h_i(S_{k-1}), & \text{otherwise.} \end{cases}$$

Then conclude $h_i(P) = \operatorname{card}\{k : \operatorname{card} G_k = i\}$. $\square$

Exercise 8.10 Finish the proof of Gram's Theorem (Section 6.4) by showing that

$$\sum_{F : 0 \leq \dim F \leq d} (-1)^{\dim F} a_F(x) = 0$$

if $x \notin P$. Suggestion: Suppose P is a d-polytope in $\mathbf{R}^d$. Each facet F_i has a unique supporting hyperplane H_i. Let H_i^+ be the closed halfspace associated with H_i containing P, and H_i^- be the opposite closed halfspace. Let F be any proper face of P. Define x to be *beyond* F (or F to be *visible* from x) if and only if there is at least one i such that $F \subset F_i$ and $x \in H_i^- \setminus H_i$. Note that $a_F(x) = 0$ if and only if F is visible from x. Now prove that the set of faces visible from x is shellable. Apply Euler's Relation (Exercise 8.6). $\square$

Definition 8.11 Let V be a finite set, and let Δ be a nonempty collection of subsets of V with the property that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. Then Δ is called an (abstract) *simplicial complex*. In particular, the empty set is in every simplicial complex. Sets of Δ of cardinality j are called *faces of dimension* $j-1$. The number of such faces is denoted f_{j-1}. Faces of dimension 0 and 1 are called *vertices* and *edges*, respectively, and faces of maximum dimension are called *facets*. The *dimension* of Δ is the dimension of its facets. The *f-vector* of a simplicial $(d-1)$-complex Δ is $(f_{-1}, \ldots, f_{d-1})$ (though sometimes we omit writing f_{-1}), and its *h-vector* $(h_0, \ldots, h_d)$ is defined via equation (5). If every face is contained in a facet, then Δ is said to be *pure*. For any finite set F in Δ let $\mathcal{F}(F)$ denote the collection of all subsets of F (including F and $\emptyset$).

Definition 8.12 We can define shellability for simplicial complexes to correspond to the situation for simplicial polytopes. Let Δ be a simplicial complex on V. Then Δ is said to be *shellable* if it is pure and its facets can be ordered $F_1, \ldots, F_n$ such that for every $k = 1, \ldots, n$ there is a face G_k in $\mathcal{F}(F_k)$ such that $\mathcal{F}(F_k) \cap (\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1}))$ is the set of all faces of F_k not containing G_k. As in Exercise 8.9, a shelling of Δ can be used to determine its h-vector.

Exercise 8.13 Let $1 \leq k \leq n$ and Consider the simplicial complex Δ on the set $V = \{1, \ldots, n\}$ such that $F \in \Delta$ iff $\operatorname{card} F \leq k$. Prove that Δ is shellable and find a "natural" shelling order for its facets.

More details on shellings can be found in [6, 16, 32, 51]. Ziegler [51] proves that not all 4-polytopes are *extendably shellable*. In particular, there exists a 4-polytope P and a collection $F_1, \ldots, F_m$ of facets of P such that $\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_m)$ is shellable with shelling order $F_1, \ldots, F_m$, but this cannot be extended to a shelling order $F_1, \ldots, F_m, F_{m+1}, \ldots, F_n$ of all of the facets of P.

9 The Upper Bound Theorem

At about the same time that polytopes were proved to be shellable, two important extremal f-vector results were also settled. What are the maximum and minimum values of $f_j(P)$ for simplicial d-polytopes with n vertices? We now know that the maxima are attained by cyclic polytopes, which we discuss in this section, and the minima by stacked polytopes, which we tackle in the next.

A good reference for this section is [32]. In fact, McMullen discovered the proof of the Upper Bound Theorem while writing this book with Shephard. Originally the intent was to report on progress in trying to solve what was then known as the Upper Bound Conjecture. See also [6, 51].

To construct cyclic polytopes, consider the *moment curve* $\{m(t) := (t, t^2, \ldots, t^d) : t \in \mathbf{R}\} \subset \mathbf{R}^d$ and choose $n \geq d+1$ distinct points $v_i = m(t_i)$ on the curve, $t_1 < \cdots < t_n$. Let $V = \{v_1, \ldots, v_n\}$ and set $C(n, d) := \operatorname{conv} V$, the convex hull of V. This is called a *cyclic polytope.*

First we will show that $C(n, d)$ is a simplicial d-polytope. Let W be the set of any d points on the moment curve and let $a_1x_1 + \cdots + a_dx_d = a_0$ be the equation of any hyperplane containing W. Then $a_1t_i + a_2t_i^2 + \cdots + a_dt_i^d = a_0$ if $v_i = m(t_i) \in W$. Therefore the polynomial $a_1t + a_2t^2 + \cdots + a_dt^d - a_0$ has at least d roots. But being nontrivial and of degree $\leq d$ it has at most d roots. Therefore there can be no other points of the moment curve on H besides W. We conclude that $C(n, d)$ is full-dimensional, and every facet contains only d vertices and hence is a simplex.

Now we prove that $C(n, d)$ has a remarkable number of lower dimensional faces.

Theorem 9.1 *Let $W \subset V$ have cardinality at most $\lfloor d/2 \rfloor$. Then $\operatorname{conv} W$ is a face of $C(n, d)$. Consequently $f_{j-1}(C(n, d)) = \binom{n}{j}$, $j = 0, \ldots, \lfloor d/2 \rfloor$.*

PROOF. Consider the polynomial

$$p(t) = \prod_{v_i \in W} (t - t_i)^2.$$

It has degree at most d, so it can be written $a_0 + a_1t + \cdots + a_dt^d$. Note that t_i is a root if $v_i \in W$ and that $p(t_i) > 0$ if $v_i \in V \setminus W$. So the vertices of V which lie on the hyperplane H whose equation is $a_1x_1 + \cdots + a_dx_d = -a_0$ are precisely the vertices in W, and H is a supporting hyperplane to $C(n, d)$. So we have a supporting hyperplane for $\operatorname{conv} W$, which is therefore a face of $C(n, d)$. □

The cyclic polytope $C(n, d)$ obviously has the maximum possible number of j-faces, $1 \le j \le \lfloor d/2 \rfloor - 1$, of any d-polytope with n vertices. Polytopes of dimension d for which $f_j = \binom{n}{j+1}$, $j = 0, \ldots, \lfloor d/2 \rfloor - 1$, are called *neighborly*. It might be supposed that $C(n, d)$ has the maximum possible number of higher dimensional faces as well. Motzkin [34] implicitly conjectured this, and McMullen[31] proved this to be the case.

We first examine the h-vector. Note that $f(C(n, d), t)$ agrees with $(1 + t)^n$ in the coefficients of t^i, $i = 0, \ldots, \lfloor d/2 \rfloor$. Knowing that h_i depends only upon $f_{-1}, \ldots, f_{i-1}$ (see Equation (5)), we have that $h(C(n, d), t) = (1 - t)^d f(C(n, d), \frac{t}{1-t})$ agrees with

$$(1 - t)^d (1 + \frac{t}{1 - t})^n = (1 - t)^{d-n} = (1 + t + t^2 + \cdots)^{n-d}$$

in the coefficients of t^i, $i = 0, \ldots, \lfloor d/2 \rfloor$. Therefore,

$$h_i(C(n, d)) = \binom{n - d + i - 1}{i}, \; i = 0, \ldots, \lfloor d/2 \rfloor$$

(verify this!). The second half of the h-vector is determined by the Dehn-Sommerville Equations.

Theorem 9.2 (Upper Bound Theorem, McMullen 1970) *If P is a convex d-polytope with n vertices, then $f_j(P) \le f_j(C(n, d))$, $j = 1, \ldots, d - 1$.*

Proof. Perturb the vertices of P slightly, if necessary, so that we can assume P is simplicial. This will not decrease any component of the f-vector and will not change the number of vertices. Since the components of the h-vector are nonnegative combinations of the components of the f-vector (Equation (6)), it suffices to show that $h_i(P) \le h_i(C(n, d))$ for all i. Because of the Dehn-Sommerville Equations, it is enough to prove $h_i(P) \le \binom{n-d+i-1}{i}$, $i = 1, \ldots, \lfloor d/2 \rfloor$.

Choose any simple d-polytope $Q \subset \mathbf{R}^d$ dual to P and recall that $h_i(P)$ by definition equals $h_i(Q)$, which equals the number of vertices of Q of indegree i whenever the edges are oriented by any sufficiently general vector $c \in \mathbf{R}^d$. Let F be any facet of Q. Then $h(F)$ can be obtained using the same vector c by simply restricting attention to the edges of Q in F.

Claim 1. $\sum_F h_i(F) = (i+1)h_{i+1}(Q) + (d-i)h_i(Q)$. Let v be any vertex of Q of indegree $i+1$. We can drop any one of the $i+1$ edges entering v and find the unique facet F containing the remaining $d-1$ edges incident to v. The vertex v will have indegree i when restricted to F. On the other hand, let v be any vertex of Q of indegree i. We can drop any one of the $d-i$ edges leaving v and find the unique facet F containing the remaining $d-1$ edges incident to v. This time the vertex v will have indegree $i+1$ when restricted to F. These two cases account for all vertices of indegree i in the sum $\sum_F h_i(F)$.

Claim 2. $\sum_F h_i(F) \le nh_i(Q)$. For, consider any facet F. We may choose a vector c so that the ordering of vertices of Q by c begins with the vertices of F (choose c to be a slight perturbation of an inner normal vector of F). It is now easy to see that with this ordering, a contribution to $h_i(F)$ gives rise to a contribution to $h_i(Q)$. Thus $h_i(F) \le h_i(Q)$, and summing over the facets of Q proves the claim.

From the two claims we can easily prove $(i+1)h_{i+1}(Q) \le (n-d+i)h_i(Q)$ from which $h_i(Q) \le \binom{n-d+i-1}{i}$ follows quickly by induction on i. □

What can be said about a d-polytope P for which $f(P) = f(C(n,d))$? Obviously P must be simplicial and neighborly. Shemer [41] has shown that there are very many non-cyclic neighborly polytopes. If d is even, every neighborly d-polytope is simplicial, but nonsimplicial neighborly d-polytopes exist when d is odd [6, 16].

Again, assume that $V = \{v_i = m(t_i),\ i = 1,\dots,n\}$, $t_1 < \cdots < t_n$, $n \ge d+1$, is the set of vertices of a cyclic polytope. Suppose W is a subset of V of cardinality d. When does W correspond to a facet of $C(n,d) = \operatorname{conv} V$? Looking back at the discussion at the beginning of this section, in which we proved $C(n,d)$ is simplicial, we observe that the polynomial

$$p(t) = \prod_{v_i \in W} (t - t_i)$$

changes sign at each of its roots, and that W corresponds to a facet if and only if the numbers $p(t_i)$ all have the same sign for $v_i \notin W$. Therefore, "between" every two nonelements of W must lie an even number of elements of W. The next theorem is immediate (see [6, 16, 32, 51]).

Theorem 9.3 (Gale's Evenness Condition) *The subset W corresponds to a facet of $C(n,d)$ if and only if for every pair $v_k, v_\ell \notin W$, $k < \ell$, the set $W \cap \{v_i : k < i < \ell\}$ has even cardinality.*

The above theorem shows that the combinatorial structure of the cyclic polytope does not depend upon the particular choice of the values t_i, $i = 1,\dots,n$. Thus, from a combinatorial point of view, we are justified in calling $C(n,d)$ *the* cyclic d-polytope with n vertices.

Example 9.4 Here is a representation of the facets of $C(8,5)$:

1	2	3	4	5	6	7	8
1	2	3	4	5			
1	2	3		5	6		
1		3	4	5	6		
1	2	3			6	7	
1		3	4		6	7	
1			4	5	6	7	
1	2	3				7	8
1		3	4			7	8
1			4	5		7	8
1				5	6	7	8
1	2	3	4				8
1	2		4	5			8
	2	3	4	5			8
1	2			5	6		8
	2	3		5	6		8
		3	4	5	6		8
1	2				6	7	8
	2	3			6	7	8
		3	4		6	7	8
			4	5	6	7	8

□

Exercise 9.5 (Ziegler [51]) Show (bijectively) that the number of ways in which $2k$ elements can be chosen from $\{1,\ldots,n\}$ in "even blocks of adjacent elements" is $\binom{n-k}{k}$. Thus, derive from Gale's evenness condition that the formula for the number of facets of $C(n,d)$ is

$$f_{d-1}(C(n,d)) = \binom{n-\lceil\frac{d}{2}\rceil}{\lfloor\frac{d}{2}\rfloor} + \binom{n-1-\lceil\frac{d-1}{2}\rceil}{\lfloor\frac{d-1}{2}\rfloor},$$

where $\lceil\cdot\rceil$ is the *least integer function*, with $\lceil\frac{k}{2}\rceil = k - \lfloor k/2\rfloor$. Here the first term corresponds to the facets for which the first block is even, and the second term corresponds to the cases where the first block is odd. Deduce

$$f_{d-1}(C(n,d)) = \begin{cases} \frac{n}{n-k}\binom{n-k}{k}, & \text{for } d=2k \text{ even,} \\ \\ 2\binom{n-k-1}{k}, & \text{for } d=2k+1 \text{ odd.} \end{cases}$$

□

As a consequence, for fixed d the number of facets of $C(n, d)$ grows like a polynomial of degree $\lfloor d/2 \rfloor$ (see [51]). Dually, this gives us an upper bound on the number of basic feasible solutions of a linear program in d variables described by n linear inequalities.

Exercise 9.6 (Ziegler [51]) Show that if a polytope is k-*neighborly* (every subset of vertices of cardinality at most k corresponds to a $(k-1)$-face), then every $(2k-1)$-face is a simplex. Conclude that if a d-polytope is $(\lfloor d/2 \rfloor + 1)$-neighborly, then it is a simplex. □

10 The Lower Bound Theorem

10.1 Stacked Polytopes

What is the least number of j-faces that a simplicial d-polytope with n vertices can have? The answer turns out to be achieved simultaneously for all j by a stacked polytope. A d-polytope $P(n,d)$ is *stacked* if either $n = d+1$ and it is a d-simplex, or else $n > d+1$ and $P(n,d)$ is obtained by building a shallow pyramid over one of the facets of some stacked polytope $P(n-1,d)$. Unlike the cyclic polytopes, not all stacked d-polytopes with n vertices are combinatorially equivalent. See [6, 16].

To calculate the f-vector of a stacked polytope $P = P(n,d)$, note first that for $n > d+1$,

$$f_j(P(n,d)) = f_j(P(n-1,d)) + \begin{cases} \binom{d}{j}, & j = 0,\ldots,d-2, \\ \\ d-1, & j = d-1. \end{cases}$$

The next exercise shows that the h-vector has a particularly simple form.

Exercise 10.1 Prove

$$f_j(P(n,d)) = \begin{cases} \binom{d+1}{j+1} + (n-d-1)\binom{d}{j}, & j = 0,\ldots,d-2, \\ \\ (d+1) + (n-d-1)(d-1), & j = d-1, \end{cases}$$

and

$$h_i(P(n,d)) = \begin{cases} 1, & i = 0 \text{ or } i = d, \\ n-d, & i = 1,\ldots,d-1. \end{cases}$$

□

Theorem 10.2 (Lower Bound Theorem, Barnette 1971, 1973) *If P is a simplicial convex d-polytope with n vertices, then $f_j(P) \geq f_j(P(n,d))$, $j = 1,\ldots,d-1$.*

The lower bound for $d = 4$ was stated by Brückner [7] in 1909 as a theorem, but his proof was later shown to be invalid (Steinitz [49]). Barnette [1] first proved the case $j = d-1$, and then the remaining cases [2]. His proof is reproduced in [6]. He also proved that if $d \geq 4$ and $f_{d-1}(P) = f_{d-1}(P(n,d))$, then P is a stacked d-polytope. Billera and Lee [4] extended this by showing that if $d \geq 4$ and $f_j(P) = f_j(P(n,d))$ for any single value of j, $j = 1,\ldots,d-1$, then P is stacked.

Even though Barnette's proof is not difficult, we will present the later proof by Kalai [17], which provides some deep insight into connections between the h-vector and other properties of simplicial polytopes.

10.2 Motion and Rigidity

The vertices and edges of a convex d-polytope provide an example of a framework. More generally, a *(bar and joint) framework* G in $\mathbf{R}^d$ is a finite collection of vertices (*joints*) $v_i \in \mathbf{R}^d$, $i \in V := \{1, \ldots, n\}$, and edges (*bars*) $v_i v_j := \text{conv}\,\{v_i, v_j\}$, $i \neq j$, $ij := (i,j) \in E \subset V \times V$. (We assume $ij \in E$ if and only if $ji \in E$.) We do not care whether the vertices are all distinct, or whether the edges coincidentally intersect each other at other than their common endpoints. Define the *dimension* of the framework to be $\dim \text{aff}\,\{v_1, \ldots, v_n\}$.

Now let $I \subseteq \mathbf{R}$ be an open interval and parameterize the vertices as $v_i(t)$ such that $v_i = v_i(0)$, $i = 1, \ldots, n$, and $\|v_i(t) - v_j(t)\|^2 = \|v_i - v_j\|^2$, $ij \in E$, for all $t \in I$; i.e., no edge is changing length. This defines a *motion* of the framework. A motion of any framework can be induced by a Euclidean motion (such as a translation or rotation) of the entire space $\mathbf{R}^d$—such induced motions are called *trivial*. A framework admitting only trivial motions is *rigid*.

Exercise 10.3 Give some examples of motions of two-dimensional frameworks. □

Since each edge-length is constant during a motion, we have

$$\begin{aligned} 0 &= \tfrac{d}{dt}[(v_i(t) - v_j(t))^T (v_i(t) - v_j(t))] \\ &= 2(v_i(t) - v_j(t))^T (v_i'(t) - v_j'(t)),\ ij \in E. \end{aligned}$$

Setting $t = 0$ and $u_i := v_i'(0)$, $i = 1, \ldots, n$, we have

$$(v_i - v_j)^T (u_i - u_j) = 0 \text{ for all } ij \in E. \tag{7}$$

By definition, any set of vectors $u_1, \ldots, u_n \in \mathbf{R}^d$ that satisfies Equation (7) is said to be an *infinitesimal motion* of the framework. It can be checked that not every infinitesimal motion is derived from a motion. The set of infinitesimal motions of a framework is a vector space and is called the *motion space* of the framework.

Exercise 10.4 Give some examples of infinitesimal motions of two-dimensional frameworks. Find some that come from motions, and some that do not. □

Exercise 10.5 Prove that $u_1, \ldots, u_n$ is an infinitesimal motion if and only if the projections of the vectors u_i and u_j onto the vector $v_i - v_j$ agree for every $ij \in E$. □

When we have an infinitesimal motion of the vertices and edges of a polytope P, then we simply say we have an *infinitesimal motion of* P.

Some infinitesimal motions of a framework are clearly trivial—for example, we may choose the vectors u_i to be all the same. To make the notion of trivial more precise, we first define an *infinitesimal motion of* $\mathbf{R}^d$ to be an assignment of a vector $u \in \mathbf{R}^d$ to *every* point $v \in \mathbf{R}^d$ (u depends upon v) such that $(v - \overline{v})^T(u - \overline{u}) = 0$ for *every* pair of points $v, \overline{v} \in \mathbf{R}^d$. An infinitesimal motion of a framework is *trivial* if it is the restriction of some infinitesimal motion of $\mathbf{R}^d$ to that framework. A framework that admits only trivial infinitesimal motions is said to be *infinitesimally rigid*. Infinitesimal rigidity implies rigidity, but a framework can be rigid without being infinitesimally rigid. But in real life, I would rather rely upon scaffolding that is infinitesimally rigid!

Exercise 10.6 Give some examples of trivial and nontrivial infinitesimal motions of two-dimensional frameworks. □

Theorem 10.7 *Let $P \subset \mathbf{R}^d$ be a d-simplex with vertices $v_1, \ldots, v_{d+1}$, and $u_1, \ldots, u_{d+1}$ be an infinitesimal motion of P. Then u_{d+1} is determined by $u_1, \ldots, u_d$.*

Proof. Let $e_i = v_i - v_{d+1}$, $i = 1, \ldots, d$. These vectors are linearly independent. The projections of u_{d+1} and u_i onto e_i must agree, $i = 1, \ldots, d$, and u_{d+1} is determined by these d projections. □

Theorem 10.8 *The dimension of the motion space of a d-simplex $P \subset \mathbf{R}^d$ is $\binom{d+1}{2}$.*

Proof. Let P have vertices $v_1, \ldots, v_{d+1} \in \mathbf{R}^d$. Choose any vector $u_1 \in \mathbf{R}^d$. There are d degrees of freedom in this choice—one for each coordinate. Choose any vector $u_2 \in \mathbf{R}^d$ such that the projections p_1^2 of u_2 and of u_1 on the vector $v_1 - v_2$ agree. There are $d-1$ degrees of freedom in this choice, since you can freely choose the component of u_2 orthogonal to p_1^2. In general, for $k = 2, \ldots, d$, choose any vector $u_k \in \mathbf{R}^d$ such that the projections p_i^k of u_k and of u_i on the vectors $v_i - v_k$ agree, $i = 1, \ldots, k-1$. There are $d-k+1$ degrees of freedom in this choice, since you can freely choose the component of u_k orthogonal to the span of $p_1^k, \ldots, p_{k-1}^k$. The resulting set of vectors $u_1, \ldots, u_{d+1}$ is an infinitesimal motion of P, all infinitesimal motions of P can be constructed in this way, and there are $d + (d-1) + \cdots + 2 + 1 + 0 = \binom{d+1}{2}$ degrees of freedom in constructing such a set of vectors. □

Theorem 10.9 *Let $P \subset \mathbf{R}^d$ be a d-simplex. Then P is infinitesimally rigid, and in fact every infinitesimal motion of P uniquely extends to an infinitesimal motion of $\mathbf{R}^d$.*

Proof. That the extension must be unique if it exists is a consequence of Theorem 10.7, for if v is any point in $\mathbf{R}^d$ and u is its associated infinitesimal motion vector, then u is uniquely determined by any subset of d vertices of P whose affine span misses v.

To show that an extension is always possible, let $v_1, \ldots, v_{d+1}$ be the vertices of P, and $u_1, \ldots, u_{d+1}$ be an infinitesimal motion of P. Then $(v_i - v_j)^T(u_i - u_j) = 0$ for all i, j. So

$$v_i^T u_i + v_j^T u_j = v_i^T u_j + v_j^T u_i \tag{8}$$

for all i, j.

Any $v \in \mathbf{R}^d$ can be uniquely written as an affine combination of $v_1, \ldots, v_{d+1}$:

$$v = \sum_{i=1}^{d+1} a_i v_i,$$

where $\sum_{i=1}^{d+1} a_i = 1$. Define

$$u = \sum_{i=1}^{d+1} a_i u_i.$$

We claim that this defines an infinitesimal motion of $\mathbf{R}^d$.

Choose $v, \overline{v} \in \mathbf{R}^d$. Assume that

$$v = \sum_{i=1}^{d+1} a_i v_i,$$

$$\overline{v} = \sum_{i=1}^{d+1} b_i v_i,$$

where $\sum_{i=1}^{d+1} a_i = \sum_{i=1}^{d+1} b_i = 1$. Let

$$u = \sum_{i=1}^{d+1} a_i u_i,$$

$$\overline{u} = \sum_{i=1}^{d+1} b_i u_i.$$

We must show that $(v - \overline{v})^T(u - \overline{u}) = 0$; i.e.,

$$\left(\sum_i a_i v_i - \sum_i b_i v_i\right)^T \left(\sum_j a_j u_j - \sum_j b_j u_j\right) = 0.$$

Equivalently, we must show

$$\sum_i \sum_j a_i a_j v_i^T u_j + \sum_i \sum_j b_i b_j v_i^T u_j = \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j b_i a_j v_i^T u_j. \tag{9}$$

Now I know there must be slicker way of doing this, but here is one way. Multiply Equation (8) by $a_i a_j$ and sum over i and j:

$$\begin{aligned}
\sum_i \sum_j a_i a_j v_i^T u_i + \sum_i \sum_j a_i a_j v_j^T u_j &= \sum_i \sum_j a_i a_j v_i^T u_j + \sum_i \sum_j a_i a_j v_j^T u_i \\
\sum_j a_j \sum_i a_i v_i^T u_i + \sum_i a_i \sum_j a_j v_j^T u_j &= \sum_i \sum_j a_i a_j v_i^T u_j + \sum_i \sum_j a_i a_j v_i^T u_j \\
\sum_i a_i v_i^T u_i + \sum_j a_j v_j^T u_j &= 2 \sum_i \sum_j a_i a_j v_i^T u_j \\
2 \sum_i a_i v_i^T u_i &= 2 \sum_i \sum_j a_i a_j v_i^T u_j \\
\sum_i a_i v_i^T u_i &= \sum_i \sum_j a_i a_j v_i^T u_j .
\end{aligned}$$

Similarly,

$$\sum_i b_i v_i^T u_i = \sum_i \sum_j b_i b_j v_i^T u_j .$$

Therefore, the left-hand side of Equation (9) equals

$$\sum_i a_i v_i^T u_i + \sum_i b_i v_i^T u_i . \tag{10}$$

Now multiply Equation (8) by $a_i b_j$ and sum over i and j:

$$\begin{aligned}
\sum_i \sum_j a_i b_j v_i^T u_i + \sum_i \sum_j a_i b_j v_j^T u_j &= \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j a_i b_j v_j^T u_i \\
\sum_i a_i v_i^T u_i + \sum_j b_j v_j^T u_j &= \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j a_j b_i v_i^T u_j \\
\sum_i a_i v_i^T u_i + \sum_i b_i v_i^T u_i &= \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j a_j b_i v_i^T u_j .
\end{aligned}$$

Therefore, the right-hand side of Equation (9) also equals (10). □

Corollary 10.10 *The dimension of the space of infinitesimal motions of* $\mathbf{R}^d$ *is* $\binom{d+1}{2}$.

Corollary 10.11 *A d-dimensional framework G in $\mathbf{R}^d$ is infinitesimally rigid if and only if its motion space has dimension $\binom{d+1}{2}$. In this case, every infinitesimal motion is determined by its restriction to any d affinely independent vertices of G.*

Corollary 10.12 *Let G and G' be two infinitesimally rigid d-dimensional frameworks in $\mathbf{R}^d$ that have d affinely independent vertices in common. Then the union of G and G' is also infinitesimally rigid.*

Given a framework G in $\mathbf{R}^d$ with vertices $v_1, \ldots, v_n$, we can write the conditions for $u_1, \ldots, u_n$ to be an infinitesimal motion in matrix form. Let $f_0 = n$ and let A be a matrix with f_1 rows, one for each edge of G, and df_0 columns, d for each vertex of G. In the row corresponding to edge ij, place the row vector $(v_j - v_i)^T$ in the d columns corresponding to v_i, and $v_i - v_j$ in the d columns corresponding to v_j. The remaining entries of A are set to 0. Let $u^T = (u_1^T, \ldots, u_n^T)$. Then $u_1, \ldots, u_n$ is an infinitesimal motion if and only if $Au = O$ (you should check this).

Theorem 10.13 *The motion space of a framework is the nullspace of its matrix A.*

10.3 Stress

The motion space provides a geometrical interpretation of the nullspace of A. What about the left nullspace? An element λ of the left nullspace assigns a number λ_{ij} $(= \lambda_{ji})$ to each edge of the framework, and the statement $\lambda^T A = O^T$ is equivalent to the equations

$$\sum_{j:ij \in E} \lambda_{ij}(v_j - v_i) = O \text{ for every vertex } i. \tag{11}$$

This can be regarded as a set of equilibrium conditions (one at each vertex) for the λ_{ij}, which may be thought of as forces or *stresses* on the edges of the framework. The left nullspace of A is called the *stress space*, and A itself is sometimes called the *stress matrix*.

Exercise 10.14 Give some examples of stresses for two-dimensional frameworks.

Putting together everything we know, we have several ways to test infinitesimal rigidity:

Theorem 10.15 *The following are equivalent for a d-dimensional framework G with f_0 vertices and f_1 edges:*

1. *The framework G is infinitesimally rigid.*

2. *The dimension of the motion space of G (the nullspace of A) is $\binom{d+1}{2}$.*

3. *The rank of A is $df_0 - \binom{d+1}{2}$.*

4. *The dimension of the stress space of G (the left nullspace of A) is $f_1 - df_0 + \binom{d+1}{2}$.*

Corollary 10.16 *Let P be a simplicial d-polytope. Then P is infinitesimally rigid if and only if the dimension of its stress space equals $g_2(P) := h_2(P) - h_1(P)$.*

Proof. From Equation (5), $h_1(P) = -df_{-1} + f_0$ and $h_2(P) = \binom{d}{d-2}f_{-1} - (d-1)f_0 + f_1$, so $h_2(P) - h_1(P) = \binom{d}{2} - (d-1)f_0 + f_1 + d - f_0 = f_1 - df_0 + \binom{d+1}{2}$. □

10.4 Infinitesimal Rigidity of Simplicial Polytopes

A good reference for this section is the paper by Roth [38]. Cauchy [9] proved that simplicial 3-polytopes are rigid. Dehn [15] used the stress matrix to prove the stronger result that these polytopes are infinitesimally rigid.

Theorem 10.17 (Dehn, 1916) *Let P be a simplicial convex 3-polytope. Then P admits only the trivial stress $\lambda_{ij} = 0$ for all edges ij.*

Proof. This proof is my slight modification of the proof presented in Roth [38].

Suppose there is a non-trivial stress. Label each edge $ij \in E$ with the sign $(+, -, 0)$ of λ_{ij}. Suppose there is a vertex v such that all edges incident to it are labeled 0. Then delete v and take the convex hull of the remaining vertices. The result cannot be two-dimensional, because it is clear that there can be no non-trivial stress on the edges of a single polygon (do you see why?). So the result is three-dimensional. If it is not simplicial, triangulate the non-simplicial faces arbitrarily, labeling the new edges 0. Repeat this procedure until you have a triangulated 3-polytope Q (possibly with some coplanar triangles) such that every vertex is incident to at least one nonzero edge. Note that every nonzero edge of Q is an edge of the original polytope P.

Now in each corner of each triangle of Q place the label 0 if the two edges meeting there are of the same sign, 1 if they are of opposite sign, and 1/2 if one is zero and the other nonzero.

Claim 1. The sum of the corner labels at each vertex v is at least four. First, because v is a vertex of P, the nonzero edges of P incident to v cannot all have the same sign. Consider now the sequence of changes in signs of just the nonzero edges of P incident to v as we circle around v. If there were only two changes in sign, the positive edges could be separated from the negative edges by a plane passing through v, since no three edges incident to v in P are coplanar. So there must be at least four changes in sign. The claim for the corner labels in Q now follows easily.

Claim 2. The sum of the three corner labels for each triangle of Q is at most two. Just check all the possibilities of the edge and corner labels for a single triangle.

Now consider the sum S of all the corner labels of Q. By Claim 1 the sum is at least $4f_0$, where f_0 is the number of vertices of Q. By Claim 2 the sum is at most $2f_2$, where f_2 is the number of triangles of Q. But $f_0 - f_1 + f_2 = 2$ by Euler's Relation, where f_1 is the number of edges of Q. Also, each triangle has three edges and each edge is in two triangles, so $3f_2 = 2f_1$. Therefore $f_2 = 2f_0 - 4$. So $4f_0 \leq S \leq 4f_0 - 8$ yields a contradiction. □

Corollary 10.18 *Simplicial convex* 3*-polytopes are infinitesimally rigid.*

PROOF. The dimension of the stress space equals 0, which equals $h_2(P) - h_1(P)$ by the Dehn-Sommerville Equations. The result follows by Corollary 10.16. □

Corollary 10.18 tells us that if we build the geometric skeleton of a simplicial 3-polytope out of bars which meet at flexible joints then the structure will be infinitesimally rigid. Similarly the structure will be infinitesimally rigid if we build the boundary of the polytope out of triangles which meet along flexible edges. However, if the structure is not convex, it might flex infinitesimally, and there are easy examples of this. Connelly [11] showed the truly remarkable fact that that there are simplicial 2-spheres immersed in $\mathbf{R}^3$ that have a finite real flex—a motion that is not just infinitesimal. Sabitov [39] proved that during such a flex the enclosed volume remains constant (the "Bellows" Theorem), a result that was extended to all triangulated orientable flexible surfaces by Connelly, Sabitov, and Walz [12]. Of course, if a convex 3-polytope is not simplicial, then its skeleton may flex (consider the cube).

Whiteley [50] extended the rigidity theorem to higher dimensions.

Theorem 10.19 (Whiteley, 1984) *Simplicial convex d-polytopes, $d \geq 3$, are infinitesimally rigid.*

PROOF. We proceed by induction on d. The result is true for $d = 3$, so assume P is a simplicial d-polytope, $d > 3$. Let v_0 be any vertex of P. Define G to be the framework consisting of all vertices and edges contained in all facets containing v_0 (the vertices and edges of the *closed star* of v_0). Let the edges of G be indexed by E. Construct Q, a vertex figure of P at v_0.

Claim 1. The stress spaces of G (regarded as a d-dimensional framework) and of Q (regarded as a $(d-1)$-dimensional framework) have the same dimension. Stresses are unaffected by Euclidean motions and by scaling of the framework, so assume without loss of generality that $v_0 = O$ and the neighbors of v_0 in G have coordinates $(v_1, a_1), \ldots, (v_m, a_m)$, with $a_1, \ldots, a_m > 1$. Assume that the hyperplane used to construct the vertex figure has equation $x_d = 1$. Hence the vertices of Q are v_i/a_i, $i = 1, \ldots, m$, regarded as a $(d-1)$-polytope in $\mathbf{R}^{d-1}$. Let λ be a stress on G. For $i = 1, \ldots, m$, the equilibrium conditions (11) imply

$$\left[\sum_{j\neq 0:ij\in E} \lambda_{ij}(v_i - v_j)\right] + \lambda_{i0}v_i = O,$$

and

$$\left[\sum_{j\neq 0:ij\in E} \lambda_{ij}(a_i - a_j)\right] + \lambda_{i0}a_i = 0.$$

Hence

$$\lambda_{i0} = -\frac{1}{a_i}\sum_{j\neq 0:ij\in E} \lambda_{ij}(a_i - a_j). \tag{12}$$

Define $\overline{\lambda}_{ij} = a_i a_j \lambda_{ij}$ for every edge $v_i v_j$ of Q. We can verify that $\overline{\lambda}$ is a stress on Q. For $i = 1, \ldots, m$,

$$\begin{aligned}
\sum_{j\neq 0:ij\in E} \overline{\lambda}_{ij}\left(\frac{v_i}{a_i} - \frac{v_j}{a_j}\right) &= \sum_{j\neq 0:ij\in E} \lambda_{ij}(a_j v_i - a_i v_j) \\
&= \sum_{j\neq 0:ij\in E} \lambda_{ij}(a_j v_i - a_i v_i) + \sum_{j\neq 0:ij\in E} \lambda_{ij}(a_i v_i - a_i v_j) \\
&= \left[\sum_{j\neq 0:ij\in E} \lambda_{ij}(a_j - a_i)\right] v_i + a_i \sum_{j\neq 0:ij\in E} \lambda_{ij}(v_i - v_j) \\
&= a_i \lambda_{i0} v_i + a_i \sum_{j\neq 0:ij\in E} \lambda_{ij}(v_i - v_j) \\
&= a_i(O) \\
&= O.
\end{aligned}$$

Conversely, starting with a stress $\overline{\lambda}$ on Q, we can reverse this process, defining $\lambda_{ij} = \overline{\lambda}_{ij}/(a_i a_j)$ for $ij \in E$, $i, j \neq 0$, and using Equation (12) to define λ_{i0} for all i. In this manner we obtain a stress for G.

Claim 2. The framework G is infinitesimally rigid. The simplicial d-polytope Q is infinitesimally rigid by induction, so by Theorem 10.15 the dimension of the stress space of Q is $f_1(Q)-(d-1)f_0(Q)+\binom{d}{2}$. Claim 1 implies that this is also the dimension of the stress of G. Hence

$$\begin{aligned} f_1(G)-df_0(G)+\binom{d+1}{2} &= (f_1(Q)+f_0(Q))-d(f_0(Q)+1)+\binom{d+1}{2} \\ &= f_1(Q)-(d-1)f_0(Q)+\binom{d}{2} \\ &= \text{the dimension of the stress space of } Q \\ &= \text{the dimension of the stress space of } G. \end{aligned}$$

Therefore G is infinitesimally rigid by Theorem 10.15.

Now consider any two adjacent vertices v and v' of G. The frameworks of their closed stars are each infinitesimally rigid by Claim 2, and share d affinely independent vertices (those on any common facet). Thus the union of these two frameworks is infinitesimally rigid by Corollary 10.12. Therefore repeated application of Theorem 10.12 implies that P is infinitesimally rigid. □

10.5 Kalai's Proof of the Lower Bound Theorem

Theorem 10.20 (Kalai, 1987) *Let P be a simplicial d-polytope with n vertices. Then $f_j(P)\geq f_j(P(n,d))$, $j=0,\dots,d-1$.*

PROOF. Since P is infinitesimally rigid, the Theorem dimension of its stress space equals $g_2:=h_2(P)-h_1(P)$. Hence this quantity is nonnegative, and so $h_2(P)\geq h_1(P)=n-d$. Therefore $f_1(P)\geq f_1(P(n,d))$.

To establish the result for higher-dimensional faces, Kalai uses the "McMullen-Perles-Walkup" (MPW) reduction. I am going to quote this proof almost verbatim from Kalai's paper, so will use his notation. Let $\phi_k(n,d):=f_k(P(n,d)$. For a simplicial d-polytope C with n vertices define $\gamma(C)=f_1(C)-\phi_1(n,d)=g_1(C)$. Thus, for $d\geq 3$, $\gamma(C)=f_1(C)-dn+\binom{d+1}{2}$ and for $d=2$, $\gamma(C)=f_1(C)-n$. Define also

$$\gamma_k(C)=f_k(C)-\phi_k(n,d).$$

Let S be any face of $\operatorname{bd} C$ with k vertices. The *link* of S in C is defined to be

$$\operatorname{lk}(S,C):=\{T : T \text{ is a face of } \operatorname{bd} C,\ T\cap S=\emptyset,\ \operatorname{conv}(T\cup S) \text{ is a face of } \operatorname{bd} C\}.$$

It is known that $\operatorname{lk}(S,C)$ is isomorphic to set of boundary faces of some $(d-k)$-polytope (take repeated vertex figures). Define

$$\gamma^k(C)=\sum\{\gamma(\operatorname{lk}(S,C)) : S\in C,\ |S|=k\}.$$

Thus, $\gamma_1(C) = \gamma^0(C) = \gamma(C)$.

Proposition 10.21 *Let C be a simplicial d-polytope, and let k, d be integers, $1 \le k \le d-1$. There are positive constants $w_i(k,d)$, $0 \le i \le k-1$, such that*

$$\gamma_k(C) = \sum_{i=0}^{k-1} w_i(k,d)\gamma^i(C). \tag{13}$$

Proof. First note that

$$(k+1)f_k(C) = \sum_{i=1}^{n} f_{k-1}(\mathrm{lk}\,(v,C)). \tag{14}$$

Put $\phi_k(n,d) = a_k(d)n + b_k(d)$. (Thus, $a_k(d) = \binom{d}{k}$ for $1 \le k \le d-2$ and $a_{d-1}(d) = d-1$.) Easy calculation gives

$$2\left(dn - \binom{d+1}{2}\right) a_{k-1}(d-1) + nb_{k-1}(d-1) = (k+1)\phi_k(n,d).$$

Let C be a simplicial d-polytope, $d \ge 3$, with n vertices $v_1, \ldots, v_n$. Assume that the degree of v_i in C is n_i (i.e., $f_0(\mathrm{lk}\,(v_i,c)) = n_i$). Note that $\sum_{i=1}^n n_i = 2f_1(C) = 2(dn - \binom{d+1}{2} + \gamma(C))$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} \phi_{k-1}(n_i, d-1) &= a_{k-1}(d-1)\sum_{i=1}^{n} n_i + nb_{k-1}(d-1)\\ &= a_{k-1}(d-1)2\left(dn - \binom{d+1}{2}\right) + 2a_{k-1}(d-1)\gamma(C) + nb_{k-1}(d-1)\\ &= (k+1)\phi_k(n,d) + 2a_{k-1}(d-1)\gamma(C).\end{aligned} \tag{15}$$

From (14) and (15) we get

$$(1+k)\gamma_k(C) = 2a_{k-1}(d-1)\gamma(C) + \sum_{i=1}^{n} \gamma_{k-1}(\mathrm{lk}\,(v_i,C)). \tag{16}$$

Repeated applications of formula (16) give (13). The value of $w_i(k,d)$ is

$$w_i(k,d) = \begin{cases} 2(a_{k-i-1}(d-i-1))/(k+1)\binom{k}{i}, & 0 \le i \le k-2,\\ 2/(k+1)k, & i = k-1.\end{cases}$$

Corollary 10.22 (The MPW-reduction) *Let $d \geq 2$ be an integer. Let C be a simplicial d-polytope with n vertices, such that $\gamma(\mathrm{lk}\,(S,C)) \geq 0$ for every face S of $\mathrm{bd}\,C$, $|S| < k$. Then*

1. *$f_k(C) \geq \phi_k(n,d)$.*
2. *If $f_k(C) = \phi_k(n,d)$ then $\gamma(C) = 0$.*

□

Exercise 10.23 Check the details of the above Proposition and Corollary. □

Kalai [17] discusses the extension of the Lower Bound Theorem to more general classes of objects.

The important insight of Kalai's proof is that $h_2 - h_1$ is nonnegative for a simplicial d-polytope, $d \geq 3$, because it *counts* something; namely, the dimension of a certain vector space. It turns out that $h_i - h_{i-1}$ is also nonnegative, $i = 3, \ldots, \lfloor d/2 \rfloor$ as well, and it is possible to generalize appropriately the notion of stress to capture this result.

10.6 The Stress Polynomial

Suppose we have a stress λ on a d-dimensional framework in $\mathbf{R}^d$ with vertices indexed by $1, \ldots, n$ and edges indexed by E. Define $\lambda_{ij} = 0$ if $i \neq j$, $ij \notin E$. Taking $\lambda_{ij} = \lambda_{ji}$ if $i \neq j$, we define

$$\lambda_{jj} := -\sum_{i:i\neq j} \lambda_{ij},\ j = 1, \ldots, n.$$

Then define

$$b(x_1, \ldots, x_n) := \sum_{i,j:i\neq j} \lambda_{ij} x_i x_j + \sum_j \frac{\lambda_{jj}}{2} x_j^2.$$

This *stress polynomial* (see Lee [28]) $b(x)$ captures the definition of stress in the following way. Let $\overline{M}$ be the $(d+1) \times n$ matrix

$$\overline{M} := \begin{bmatrix} v_1 & \cdots & v_n \\ 1 & \cdots & 1 \end{bmatrix}.$$

Theorem 10.24 *λ is a stress if and only if $\overline{M}\nabla b = O$.*

In this theorem, $\nabla b(x) := (\frac{\partial}{\partial x_1} b(x), \ldots, \frac{\partial}{\partial x_n} b(x))$, and we are regarding $\overline{M}\nabla b$ as a member of $(\mathbf{R}[x_1, \ldots, x_n])^{d+1}$.

Proof. Starting with the equilibrium conditions for stress,

$$\sum_{i:i \neq j} \lambda_{ij}(v_i - v_j) = O, \ j = 1, \ldots, n,$$

$$\sum_{i:i \neq j} \lambda_{ij} v_i + \left(- \sum_{i:i \neq j} \lambda_{ij} \right) v_j = O, \ j = 1, \ldots, n,$$

$$\sum_{i:i \neq j} \lambda_{ij} v_i + \lambda_{jj} v_j = O, \ j = 1, \ldots, n.$$

$$\sum_{i=1}^{n} \lambda_{ij} v_i = O, \ j = 1, \ldots, n.$$

Also, obviously,

$$\sum_{i:i \neq j} \lambda_{ij} + \lambda_{jj} = 0, \ j = 1, \ldots, n,$$

$$\sum_{i=1}^{n} \lambda_{ij} = 0, \ j = 1, \ldots, n.$$

Also,

$$\frac{\partial}{\partial x_i} b(x) = \sum_{j=1}^{n} \lambda_{ij} x_j, \ i = 1, \ldots, n.$$

Therefore

$$\begin{aligned} \overline{M}\nabla b &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_{ij} x_j \right) \begin{bmatrix} v_i \\ 1 \end{bmatrix} \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \lambda_{ij} \begin{bmatrix} v_i \\ 1 \end{bmatrix} \right) x_j \\ &= \begin{bmatrix} O \\ 0 \end{bmatrix} x_j \\ &= \begin{bmatrix} O \\ 0 \end{bmatrix}. \end{aligned}$$

□

Some remarks:

1. Every nonzero coefficient of the stress polynomial is associated naturally to a certain face of the framework.

2. The coefficients of the square-free terms uniquely determine the coefficients of the x_i^2 terms.

3. If we define the matrix
$$M := \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]$$
then the condition of Theorem 10.24 can be written as $M\nabla b(x) = O$ and $\omega b(x) = 0$, where
$$\omega := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}.$$

11 Simplicial Complexes

Since boundaries of simplicial polytopes provide examples of simplicial complexes, we now study what we can determine about f-vectors and h-vectors of various classes of simplicial complexes. Stanley's book [48] is a good source of material for this section, and provides further references.

Recall our definitions:

Let V be a finite set. An *(abstract) simplicial complex* Δ is a nonempty collection of subsets of V such that $F \subset G \in \Delta$ implies $F \in \Delta$. In particular, $\emptyset \in \Delta$. For $F \in \Delta$ we say F is a *face* of Δ and the *dimension* of F, $\dim F$, equals $\operatorname{card}(F) - 1$. We define $\dim \Delta := \max\{\dim F : F \in \Delta\}$ and refer to a simplicial complex of dimension $d-1$ as a *simplicial* $(d-1)$*-complex.* Faces of dimension 0, 1, $d-2$, and $d-1$ are called *vertices* *edges*, *subfacets* or *ridges*, and *facets* of Δ, respectively. For simplicial $(d-1)$-complex Δ we define $f_j(\Delta)$ to be the number of j-dimensional (j*-faces*) of Δ, and its f*-vector* to be $f(\Delta) := (f_0(\Delta), f_1(\Delta), \ldots, f_{d-1}(\Delta))$, and then use the same equation (5) for simplicial d-polytopes to define the h*-vector* of Δ.

Exercise 11.1 Suppose Δ is a simplicial complex on $V = \{1, \ldots, n\}$. Prove that there exists a positive integer e and points $v_1, \ldots, v_n \in \mathbf{R}^e$ such that $\operatorname{conv}\{v_i : i \in F\} \cap \operatorname{conv}\{v_i : i \in G\} = \operatorname{conv}\{v_i : i \in F \cap G\}$. In this way we can realize any simplicial complex geometrically. $\square$

11.1 The Kruskal-Katona Theorem

For positive integers a and i, a can be expressed uniquely in the form

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. This is called the *$i$-canonical representation of a.*

Exercise 11.2 Prove that the i-canonical representation exists and is unique. $\square$

From this representation define

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}$$

where $\binom{k}{\ell} = 0$ if $\ell > k$. Define also $0^{(0)} = 0$.

Kruskal [21] characterized the f-vectors of simplicial complexes in 1963. Katona [18] found a shorter proof in 1968. The theorem is also a consequence of the generalization by Clements and Lindström [10].

Theorem 11.3 (Kruskal-Katona) *The vector $(f_{-1}, f_0, \ldots, f_{d-1})$ of positive integers is the f-vector of some simplicial $(d-1)$-dimensional complex Δ if and only if*

1. $f_{-1} = 1$, *and*
2. $f_j \leq f_{j-1}^{(j)}$, $j = 1, 2, \ldots, d-1$.

PROOF. (Sketch.)

Sufficiency: Let $V = \{1, 2, \ldots\}$. Let $V^i = \{F \subseteq V : |F| = i\}$. Order the sets in V^i *reverse lexicographically*. That is, for $F, G \in V^i$, $F \neq G$, define $F < G$ if there exists a k such that $k \notin F$, $k \in G$, and $i \in F$ if and only if $i \in G$ for all $i > k$. (This might more properly be referred to as *co-lex* order.) For all j choose the first f_{j-1} sets of V^j. The conditions will force the resulting collection to be a simplicial complex.

Example:

1	6	13	10
$\underline{\emptyset}$	1	12	123
	2	13	124
	3	23	134
	4	14	234
	5	24	125
	$\underline{6}$	34	135
		15	235
		25	145
		35	245
		45	$\underline{345}$
		16	126
		26	136
		$\underline{36}$	236
		46	146
		56	246
			346
			156
			256
			356
			456

Necessity: Given simplicial complex Δ. By application of a certain "shifting" or "compression" operation, transform it to a reverse lexicographic simplicial complex with the same f-vector. Then verify that the conditions must hold. $\square$

Corollary 11.4 *f-vectors of simplicial d-polytopes must satisfy the Kruskal-Katonal conditions.*

11.2 Order Ideals of Monomials

We will soon see that understanding how to count monomials will help in investigating h-vectors of certain simplicial complexes. Let X be the finite set $\{x_1, \ldots, x_n\}$. An *order ideal of monomials* is a nonempty set M of monomials $x_1^{b_1} \cdots x_n^{b_n}$ in the variables x_i such that $m|m' \in M$ implies $m \in M$. In particular, $1 = x_1^0 \cdots x_n^0 \in M$. Let $h_i(M)$ be the number of monomials in M of degree i. The sequence $h = (h_0(M), h_1(M), \ldots)$ is called an *M-sequence*, or an *M-vector* if it terminates $(h_0, \ldots, h_d)$ for some d.

For positive a and i, use the i-canonical representation of a to define

$$a^{\langle i \rangle} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j+1}.$$

Define also $0^{\langle i \rangle} = 0$.

Stanley [46] (see also [44, 45, 48]) proved the following characterization of M-sequences of order ideals of monomials, which is analogous to the Kruskal-Katona Theorem, as one piece of a much larger program in which he established, elucidated and exploited new connections between combinatorics and commutative algebra.

Theorem 11.5 (Stanley) *$(h_0, h_1, h_2, \ldots)$, a sequence of nonnegative integers, is an M-sequence if and only if*

1. *$h_0 = 1$, and*
2. *$h_{i+1} \le h_i^{\langle i \rangle}$, $i = 1, 2, 3, \ldots$.*

Proof. (Sketch.)

Sufficiency: Let M^i be the set of all monomials of degree i. Order the monomials in M^i *reverse lexicographically*. That is, for $m, m' \in M^i$, $m \ne m'$, $m = x_1^{b_1} \cdots x_n^{b_n}$, $m' = x_1^{b'_1} \cdots x_n^{b'_n}$, we say $m < m'$ if there is some k such that $b_k < b'_k$ and $b_i = b'_i$ for all $i > k$. For all i choose the first h_i monomials of M^i. The conditions will force the resulting collection to be an order ideal of monomials.

Example:

1	3	4	2
$\underline{1}$	x_1	x_1^2	x_1^3
	x_2	x_1x_2	$\underline{x_1^2x_2}$
	$\underline{x_3}$	x_2^2	$x_1x_2^2$
		$\underline{x_1x_3}$	x_2^3
		x_2x_3	$x_1^2x_3$
		x_3^2	$x_1x_2x_3$
			$x_2^2x_3$
			$x_1x_3^2$
			$x_2x_3^2$
			x_3^3

Complete details can be found in Billera and Lee [4].

Necessity: Given an order ideal of monomials, "shift" or "compress" it to a reverse lexicographic order ideal with the same M-sequence. Then verify that the conditions must hold. The fact that the compression technique results in an order ideal of monomials is due to Macaulay [30] (hence Stanley's choice of "M" in "M-sequence"). Clements and Lindström [10] provide a more accessible proof of a generalization of Macaulay's theorem. □

11.3 Shellable Simplicial Complexes

Let Δ be a simplicial $(d-1)$-complex. We say that Δ is *shellable* if it is pure (every face of Δ is contained in a facet), with the property that the facets can be ordered $F_1, \ldots, F_m$ such that for $k = 2, \ldots, m$ there is a unique minimal nonempty face G_k of $\mathcal{F}(F_k)$ that is not in $S_{k-1} := \mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1})$. See Exercise 8.9, in which we conclude that for every k,

$$h_i(S_k) = \begin{cases} h_i(S_{k-1}) + 1, & i = f_0(G_k), \\ h_i(S_{k-1}), & \text{otherwise.} \end{cases}$$

Stanley [45, 46] stated the following theorem:

Theorem 11.6 (Stanley) $(h_0, \ldots, h_d) \in \mathbf{Z}_+^{d+1}$ *is the h-vector of some shellable simplicial $(d-1)$-complex if and only if it is an M-vector.*

PROOF. We will sketch Stanley's construction for sufficiency, leaving the necessity of the conditions for later. I think the construction first appeared in Lee [26]. This type of construction was the core of the combinatorial portion of the proof in [4]. See also [27] for a slight generalization.

Let $V = \{1, 2, 3, \ldots\}$. Let V^i be the collection of all subsets F of V of cardinality d such that $1, \ldots, d-i \in F$ but $d-i+1 \notin F$. For all i choose the first h_i sets in V^i, using reverse lexicographic order. We claim that these are the facets of Δ, and they are shellable in reverse lexicographic order. Further, if a chosen F is in V^i then it contributes to $h_i(\Delta)$ during the shelling.

Associate with each facet $F = \{i_1, \ldots, i_d\}$ $(i_1 < \cdots < i_d)$ the monomial $m(F)$ $x_{i_1-1}x_{i_2-2}\cdots x_{i_d-d}$, where we interpret $x_0 = 1$. Then $F \in V^i$ if and only if $\deg m(F) = i$. By the proof of Theorem 11.5, the selected monomials will form an order ideal that is also closed (within each degree) under reverse lexicographic order. We call such a collection of monomials a *lexicographic order ideal*.

Example: $d = 3$, $h = (1, 3, 4, 2)$. Chosen facets are marked with an asterisk.

$m(F)$	i	1	2	3	4	5	6
1	0^*	1	2	3			
x_1	1^*	1	2		4		
x_1^2	2^*	1		3	4		
x_1^3	3^*		2	3	4		
x_2	1^*	1	2			5	
x_1x_2	2^*	1		3		5	
$x_1^2x_2$	3^*		2	3		5	
x_2^2	2^*	1			4	5	
$x_1x_2^2$	3		2		4	5	
x_2^3	3			3	4	5	
x_3	1^*	1	2				6
x_1x_3	2^*	1		3			6
$x_1^2x_3$	3		2	3			6
x_2x_3	2	1			4		6
$x_1x_2x_3$	3		2		4		6
$x_2^2x_3$	3			3	4		6
x_3^2	2	1				5	6
$x_1x_3^2$	3		2			5	6
$x_2x_3^2$	3			3		5	6
x_3^3	3				4	5	6

Let $F = \{i_1, \ldots, i_d\}$ be a facet of V^i in Δ. Choose $G = F \setminus \{1, \ldots, d-i\}$. It is not hard to prove that no facet preceding F in reverse lexicographic order contains G.

Choose $j > d-i$. Find $k_j := \max\{k \notin F : k < i_j\}$ and define $F_j := (F \setminus \{i_j\}) \cup \{k_j\}$. Obviously F and F_j are neighbors, i.e., share $d-1$ elements. Knowing that the monomials associated with the facets in Δ form a lexicographic order ideal, we can also verify that $F_j \in \Delta$ for every j.

From the above analysis it is possible to conclude that G is the unique minimal new face of F added to Δ when F is shelled onto the preceding facets of Δ. $\square$

12 The Stanley-Reisner Ring

12.1 Overview

To finish the proof of the previous section and show that the h-vector of a shellable simplicial complex Δ is an M-vector, we need to show how to construct a suitable order ideal of monomials from Δ. This is facilitated by certain algebraic tools developed by Stanley. A good general reference is [48].

Let Δ be a simplicial $(d-1)$-complex with n vertices $1,\ldots,n$. Consider the polynomial ring $R = \mathbf{R}[x_1,\ldots,x_n]$. There is a natural grading of $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ by degree, where R_i consists of only those polynomials, each of whose terms have degree exactly i. For a monomial $m = x_1^{a_1}\cdots x_n^{a_n}$ in R we define the *support* of m to be $\mathrm{supp}(m) = \{i : a_i > 0\}$. Let I be the ideal of R generated by monomials m such that $\mathrm{supp}(m) \notin \Delta$. The *Stanley-Reisner ring* or *face ring* of Δ is $A_\Delta := R/I$. Informally, we do calculations as in R but set any monomial to zero whose support does not correspond to a face.

The ring A_Δ is also graded $A_\Delta = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ by degree. We will see that $\sum \dim A_i t^i = f(\Delta, \frac{t}{1-t})$. Stanley proved that if Δ is shellable, then there exist d elements $\theta_1,\ldots,\theta_d \in A_1$ (a *homogeneous system of parameters*) such that θ_i is not a zero-divisor in $A_\Delta/(\theta_1,\ldots,\theta_{i-1})$, $i=1,\ldots,d$. Let $B = A_\Delta/(\theta_1,\ldots,\theta_d) = B_0 \oplus B_1 \oplus \cdots \oplus B_d$. Then $\sum \dim B_i t^i = (1-t)^d f(\Delta, \frac{t}{1-t}) = h(\Delta, t)$. So $\dim B_i = h_i$, $i = 0,\ldots,d$. Macaulay [30] proved that there exists a basis for B as an $\mathbf{R}$-vector space that is an order ideal of monomials. Theorem 11.6 then follows immediately from Theorem 11.5.

The existence of the θ_i means that A_Δ is *Cohen-Macaulay* and as a consequence $h(\Delta)$ is an M-vector. Reisner [37] characterized the class of Cohen-Macaulay complexes, those simplicial complexes Δ for which A_Δ is a Cohen-Macaulay ring:

Theorem 12.1 (Reisner) *A simplicial complex Δ is Cohen-Macaulay if and only if for all $F \in \Delta$, $\dim \tilde{H}_i(\mathrm{lk}_\Delta F, \mathbf{R} = 0$ when $i < \dim \mathrm{lk}_\Delta F$.*

In particular, simplicial complexes that are topological balls (*simplicial balls*) and spheres (*simplicial spheres*), whether shellable or not, are Cohen-Macaulay.

12.2 Shellable Simplicial Complexes are Cohen-Macaulay

Let Δ be a simplicial $(d-1)$-complex with n vertices $1,\ldots,n$, and consider the ring $A_\Delta = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$.

Theorem 12.2 (Stanley) *The dimension of A_ℓ as a vector space over* $\mathbf{R}$ *is $H_\ell(\Delta)$, where*

$$H_\ell(\Delta) = \begin{cases} 1, & \ell = 0, \\ \displaystyle\sum_{j=0}^{\ell-1} f_j(\Delta)\binom{\ell-1}{j}, & \ell > 0, \end{cases}$$

(taking $f_j(\Delta) = 0$ if $j \geq d$).

Proof. We need to show that the number of nonzero monomials of degree ℓ in A_Δ is $H_\ell(\Delta)$. Let F be a face of dimension j (hence cardinality $j+1$). By Exercise 7.10 the number of monomials m of degree ℓ such that $\operatorname{supp}(m) = \operatorname{supp}(F)$ is $\binom{\ell-1}{j}$. The result now follows easily. □

Let T be any $d \times n$ matrix such that every $d \times d$ submatrix associated with a facet of Δ is invertible. If Δ happens to be the boundary complex of a simplicial d-polytope $P \subset \mathbf{R}^d$ containing the origin in its interior, then we can take T to be the matrix whose columns are the coordinates of the vertices of P. For $i = 1, \ldots, d$ define $\theta_i = t_{i1}x_1 + \cdots + t_{in}x_n \in A_1$. That is to say, θ_i is a linear expression whose coefficients can be read off from row i of T.

Theorem 12.3 (Reisner-Stanley) *There exist monomials $\eta_1, \ldots, \eta_m \in A$ such that every member y of A has a unique representation of the form $y = \sum_{i=1}^{m} p_i\eta_i$, where the p_i are polynomials in the θ_i.*

Proof. We sketch the proof of Kind and Kleinschmidt [19]. Let $F_1, \ldots, F_m$ be a shelling of the facets of Δ and define S_j to be the collection of all faces of Δ in $F_1, \ldots, F_j$, $j = 1, \ldots, m$. Define the Stanley-Reisner ring A_{S_j} of S_j in the natural way. We will prove by induction on j that A_{S_j} has the property of the theorem.

For facet F_j, the columns in T corresponding to the vertices of F_j determine a $d \times d$ submatrix U of T. Multiply T on the left by U^{-1} to get the matrix T'. For $i = 1, \ldots, d$ define $\theta'_i := t'_{i1}x_1 + \cdots + t'_{in}x_n \in A_1$. Then the θ'_i are linear combinations of the θ_i and vice versa since the relations are invertible.

First suppose $j = 1$. For convenience, suppose F_1 contains the vertices $1, \ldots, d$. Then $x_i = 0$, $i = d+1, \ldots, n$ and $\theta'_i = x_i$, $i = 1, \ldots, d$. The elements of A_{S_1} are precisely the polynomials in the variables $x_1, \ldots, x_d$. Since $x_1^{a_1} \cdots x_d^{a_d} = \theta'^{a_1}_1 \cdots \theta'^{a_d}_d$, choosing $\eta_1 = 1$ we can see that every member y of A_{S_1} has a representation of the form $y = p'_1\eta_1$ where p'_1 is a polynomial in the θ'_i. Transforming back to the θ_i, $p'_1(\theta'_1, \ldots, \theta'_d) = p_1(\theta_1, \ldots, \theta_d)$, a polynomial in the θ_i. To show that the representation is unique, suppose $p_1(\theta_1, \ldots, \theta_d) = 0$ for some polynomial p_1 in the θ_i. Transforming the θ_i to θ'_i, we have a polynomial p'_1 in the $\theta'_i = x_i$ which equals 0. Therefore p'_1, and hence p_1, must be the zero polynomial.

Now suppose $j > 1$. Let G_j be the unique minimal face of F_j that is not present in S_{j-1}, and let $k := \operatorname{card} G_j$. For convenience, assume F_j contains the vertices $1, \ldots, d$ and G_j contains the vertices $1, \ldots, k$. Let $\eta_j := x_1 \cdots x_k$.

Consider any nonzero monomial m in A_{S_j} that is divisible by η_j. Then the support of m contains G_j and can therefore consist only of variables from among $x_1, \ldots, x_d$ since all faces in S_j containing G_j are subsets of F_j. Then $m = m'\eta_j$ where $m' = x_1^{a_1} \cdots x_d^{a_d}$. It is now easy to check that $m'\eta_j = {\theta'}_1^{a_1} \cdots {\theta'}_d^{a_d}\eta_j$ since upon expanding, all monomials are divisible by η_j and those containing variables other than $x_1, \ldots, x_d$ are zero in A_{S_j}. From this, transforming the θ'_i to the θ_i, we can see that m can be expressed in the form $p_j\eta_j$, where p_j is a polynomial in the θ_i. Since we can handle monomials divisible by η_j, it is now easy to see that any $y \in A_{S_j}$ that is divisible by η_j can be expressed as a product of a polynomial in the θ_i and the monomial η_j.

Now consider any $y \in A_{S_j}$ such that no monomial in y is divisible by η_j. Then, regarding y as a member of $A_{S_{j-1}}$, $y = \sum_{i=1}^{j-1} p_i\eta_i$. But this may no longer be true in A_{S_j} since after expanding the sum there may be some monomials left over that are divisible by η_j, which were zero in $A_{S_{j-1}}$, but not in A_{S_j}. So $y = \sum_{i=1}^{j-1} p_i\eta_i + w$, where w is divisible by η_j. Now find a representation for w as in the preceding paragraph.

It remains to show that the representations are unique. Assume that $\sum_{i=1}^{j} p_i\eta_i = 0$. Setting all terms divisible by η_j equal to zero, it must be the case that $\sum_{i=1}^{j-1} p_i\eta_i = 0$ in $A_{S_{j-1}}$. So each of the polynomials $p_1, \ldots, p_{j-1}$ is the zero polynomial by induction. Hence $p_j\eta_j = 0$ in A_{S_j}. Transforming the θ_i to the θ'_i, we have $p'_j\eta_j = 0$ for some polynomial p'_j in the θ'_i. But for each term in the expansion, ${\theta'}_1^{a_1} \cdots {\theta'}_d^{a_d}\eta_j = x_1^{a_1} \cdots x_d^{a_d}\eta_j$, from which one readily sees that p'_j must be the zero polynomial. Transforming the θ'_i back to the θ_i, p_j must be the zero polynomial. □

The proof given above shows that A_Δ is a free module over the ring $\mathbf{R}[\theta_1, \ldots, \theta_d]$ and that $\eta_1, \ldots, \eta_m$ is a monomial basis. Further, there are exactly $h_i(\Delta)$ elements of the basis of degree i. We can construct another monomial basis in the following way. Order the monomials lexicographically by defining $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if $a_1 = b_1, \ldots, a_j = b_j$ but $a_{j+1} < b_{j+1}$. Now choose a basis in a greedy fashion by letting $\eta_1 := 1$ and η_j be the first monomial lexicographically that you cannot represent using $\eta_1, \ldots, \eta_{j-1}$. Call the resulting basis M. It will still have exactly $h_i(\Delta)$ elements of degree i.

Theorem 12.4 *The basis M is an order ideal of monomials.*

PROOF. We need to show that if η is in M then so are all its divisors. For suppose not. Then there is a divisor m of η that was not chosen. It was considered before η because $m < \eta$ ($m = m'\eta_i < m'm = \eta$). It was rejected because $m = \sum p_i \eta_i$ for the η_i in M that are less than m. But $\eta = mm'$ for some monomial m'. So $\eta = \sum p_i m' \eta_i$. But $m'\eta_i < \eta$ for each i, so each of these can be expressed in terms of the η_j in M that are less than η. Hence η itself can be expressed in terms of the preceding η_j in M, contradicting the fact that η is a basis element. $\square$

Corollary 12.5 *If Δ is a Cohen-Macaulay simplicial $(d-1)$-complex with n vertices, then*

$$h_i \leq \binom{n-d+i-1}{i}, \ i = 1, \ldots, d.$$

PROOF. There are precisely $h_1(\Delta) = n - d$ monomials of degree one in M. So by Exercise 7.9 there can be no more than $\binom{n-d+i-1}{i}$ monomials of degree i in M. Therefore $h_i \leq \binom{n-d+i-1}{i}$. $\square$

This provides a new proof of the Upper Bound Theorem for simplicial d-polytopes. As mentioned above, triangulated $(d-1)$-spheres S are also Cohen-Macaulay. It is a simpler fact to prove that the Dehn-Sommerville equations are also satisfied. Using Theorem 12.2 one can show that there must be $h_i(S)$ monomials of degree i in a monomial basis for B. This is done by realizing that a basis for A as a vector space over $\mathbf{R}$ is obtained by multiplying monomials in the θ_i by elements in the basis M for B. From this one immediately has

Theorem 12.6 (Upper Bound Theorem for Spheres, Stanley) *Let S be a triangulated $(d-1)$-sphere with n vertices. Then $h_i(S) \leq h_i(C(n,d))$, $i = 0, \ldots, d$, and $f_j(S) \leq f_j(C(n,d))$, $j = 0, \ldots, d-1$.*

References

[1] D. W. Barnette, The minimum number of vertices of a simple polytope, *Israel J. Math.* **10** (1971) 121–125.

[2] D. W. Barnette. A proof of the lower bound conjecture for convex polytopes, *Pacific J. Math.* **46** (1973) 349–354.

[3] N. L. Biggs, E. K. Lloyd, and R. J. Wilson, *Graph Theory, 1736–1936*, second edition, Oxford, 1986.

[4] L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, *J. Combin. Theory Ser. A* **31** (1981) 237–255.

[5] G. Blind and R. Blind, The semiregular polytopes, *Comment. Math. Helv.* **66** (1991) 150–154.

[6] A. Brøndsted, *An Introduction to Convex Polytopes*, Springer, New York, 1983.

[7] M. Brückner, Über die Ableitung der allgemeinen Polytope und die nach Isomorphismus verschiedenen Typen der allgemeinen Achtzelle, *Verh. Konink. Acad. Wetensch. Sect. I* **10** (1909).

[8] H. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.* **29** (1972) 197–205.

[9] A. L. Cauchy, Sur les polygones et les polyèdres, *J. Ecole Polytech.* **9** (1813) 87–98.

[10] G. F. Clements and B. Lindström, A generalization of a combinatorial theorem of Macaulay, *J. Combinatorial Theory* **7** (1969) 230–238.

[11] R. Connelly, A counterexample to the rigidity conjecture for polyhedra, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977) 333–338.

[12] R. Connelly, I. K. Sabitov, and A. Walz, The bellows conjecture, *Beiträge Algebra Geom.* **38** (1997) 1–10.

[13] P. R. Cromwell, *Polyhedra*, Cambridge, 1997.

[14] M. Dehn, Die Eulersche Formel in Zusammenhang mit dem Inhalt in der nicht-Euklidischen Geometrie, *Math. Ann.* **61** (1905) 561–586.

[15] M. Dehn, Über die Starrheit konvexer Polyeder, *Math. Ann.* **77** (1916) 466–473.

[16] B. Grünbaum, *Convex Polytopes*, second edition, Springer, 2003.

[17] G. Kalai, Rigidity and the lower bound theorem I, *Invent. Math.* **88** (1987) 125–151.

[18] G. O. H. Katona, A theorem of finite sets, in *Theory of graphs (Proc. Colloq., Tihany, 1966)*, 187–207, Academic Press, New York, 1968.

[19] B. Kind and P. Kleinschmidt, Schälbare Cohen-Macauley-Komplexe und ihre Parametrisierung, *Math. Z.* **167** (1979) 173–179.

[20] V. Klee, A combinatorial analogue of Poincaré's duality theorem, *Canad. J. Math.* **16** (1964) 517–531.

[21] J. B. Kruskal, The number of simplices in a complex, in *Mathematical optimization techniques*, 251–278, Univ. of California Press, Berkeley, Calif., 1963.

[22] I. Lakatos, *Proofs and Refutations*, Cambridge, 1976.

[23] J. F. Lawrence, Polytope volume computation, *Math. Comp.* **57** (1991) 259–271.

[24] J. F. Lawrence, Rational-function-valued valuations on polyhedra, in *Discrete and Computational Geometry (New Brunswick, NJ, 1989/1990)*, Amer. Math. Soc., Providence, RI, 1991, pages 199–208.

[25] J. F. Lawrence, A short proof of Euler's relation for convex polytopes, *Canad. Math. Bull.* **40** (1997) 471–474.

[26] C. W. Lee. *Counting the Faces of Simplicial Convex Polytopes*, PhD thesis, Cornell University, Ithaca, New York, 1981.

[27] C. W. Lee. Two combinatorial properties of a class of simplicial polytopes, *Israel J. Math.* **47** (1984) 261–269.

[28] C. W. Lee, P.L.-spheres, convex polytopes, and stress, *Discrete Comput. Geom.* **15** (1996) 389–421.

[29] J. Lee, *A First Course in Combinatorial Optimization*, Cambridge, 2004.

[30] F. S. Macaulay, Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **26** (1927) 531–555.

[31] P. McMullen, The maximum numbers of faces of a convex polytope, *Mathematika* **17** (1970) 179–184.

[32] P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, Cambridge, London, 1971.

[33] P. McMullen and D. W. Walkup, A generalized lower-bound conjecture for simplicial polytopes, *Mathematika* **18** 1971 264–273.

[34] T. S. Motzkin, Comonotone curves and polyhedra, *Bull. Amer. Math. Soc.* **63** (1957) 35.

[35] H. Poincaré, Sur la généralisation d'un théorème d'euler relatif aux polyèdres, *C. R. Acad. Sci. Paris* **177** (1893) 144–145.

[36] H. Poincaré, Complément à l'analysis situs, *Rend. Circ. Mat. Palermo* **13** (1899) 285–343.

[37] G. A. Reisner. Cohen-Macaulay quotients of polynomial rings, *Advances in Math.* **21** (1976) 30–49.

[38] B. Roth, Rigid and flexible frameworks, *Amer. Math. Monthly* **88** (1981) 6–21.

[39] I. K. Sabitov, On the problem of the invariance of the volume of a deformable polyhedron, *Uspekhi Mat. Nauk* **50** (1995) 223–224.

[40] L. Schläfli, Theorie der vielfachen Kontinuität, *Denkschr. Schweiz. naturf. Ges.* **38** (1901) 1–237.

[41] I. Shemer, Neighborly polytopes, *Israel J. Math.* **43** (1982) 291–314.

[42] D. M.-Y. Sommerville, The relations connecting the angle-sums and volume of a polytope in space of n dimensions *Proc. Roy. Soc. London Ser. A* **115** (1927) 103–119.

[43] R. P. Stanley, Cohen-Macaulay rings and constructible polytopes, *Bull. Amer. Math. Soc.* **81** (1975) 133–135.

[44] R. P. Stanley, The upper bound conjecture and Cohen-Macaulay rings, *Studies in Appl. Math.* **54** (1975) 135–142.

[45] R. P. Stanley, Cohen-Macaulay complexes, In *Higher Combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976)*, pages 51–62. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31. Reidel, Dordrecht, 1977.

[46] R. P. Stanley, Hilbert functions of graded algebras. *Advances in Math.* **28** (1978) 57–83.

[47] R. P. Stanley, The number of faces of a simplicial convex polytope, *Adv. in Math.* **35** (1980) 236–238.

[48] R. P. Stanley, *Combinatorics and Commutative Algebra*, second edition, Birkhäuser, Boston, 1996.

[49] E.Steinitz, *Polyeder und Raumeinteilungen*, volume 3, pages 1–139, 1922.

[50] W. Whiteley, Infinitesimally rigid polyhedra. I. Statics of frameworks, *Trans. Amer. Math. Soc.* **285** (1984) 431–465.

[51] G. M. Ziegler, *Lectures on Polytopes*, Springer, 2008.

Index