Polyhedra in Mathematics Education

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MathFest — August 2011

1. Math Education

2. Past

3. Present

4. Opportunities

Math Education

- K–12 students
- Teachers
- Research on learning and teaching

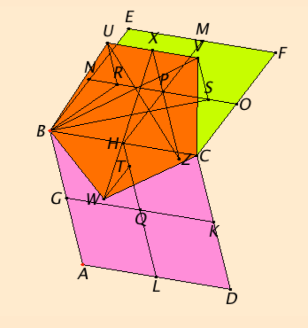
The Past

Euclid's *Elements*

Books XI–XII: Solids, including their properties and "volumes."

Book XIII: Construction of the Platonic solids.

http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

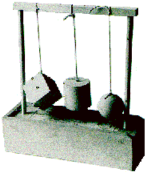


Euclid's Elements

Boyer: "The Elements of Euclid not only was the earliest major Greek mathematical work to come down to us, but also the most influential textbook of all times. . . . The first printed versions of the Elements appeared at Venice in 1482, one of the very earliest of mathematical books to be set in type; it has been estimated that since then at least a thousand editions have been published."

Friedrich Froebel

Friedrich Froebel, born in Germany 1782, founder of "Kingergarten." Developed a set of "gifts" for children, one of which suggested an investigation of properties of solids.



http://froebelweb.tripod.com/pix/gift2.gif

Comments by Thomas Banchoff: http://www.math.brown.edu/~banchoff/STG/ma8/papers/atarbox/froebel.html

Article by Richard Thiessen: www.aimsedu.org/resources/articles/files/FroebelGift.pdf

Geometry Standards

"Geometric education should begin in the kindergarten or primary school, where the child should acquire familiarity through the senses with simple geometric forms, by inspecting, drawing, modelling, and measuring them, and noting their more obvious relations. This study should be followed, in the grammar school, by systematic instruction in concrete (or observational) geometry, of which geometric drawing should form a part. Such instruction should include the main facts of plane and solid geometry, treated as matters of observation, and not as exercises in logical deduction, without however necessarily excluding the beginnings of deductive proof as soon as the pupil is ready for them. . . ."

Geometry Standards
(Winslow Homer)

Geometry Standards

"The requirement in geometry embraces the following topics: the general properties of plane rectilinear figures; the circle and the measure of angles; similar polygons; areas; regular polygons, and the measure of the circle; the relations of planes and lines in space; the properties and measure of prisms, pyramids, cylinders, and cones; the sphere and the spherical triangle."

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—*The Harvard University Catalog*, 1898–99, geometry entrance requirements for admission examination.

Old Textbooks

- Webster Wells, *Essentials of Geometry*, 1899.
- George Wentworth, *Plane and Solid Geometry*, 1899.
- Edward Rutledge Robbins, *Solid Geometry*, 1907.
- C.A. Hart and Daniel D. Feldman, *Solid Geometry*, 1912.

Primarily synthetic geometry of polyhedra, with many results on congruence, similarity, volume, surface area, and the Platonic solids that have now dropped out of the high school curricula. Few, if any, references to Euler's formula.

Old Textbooks

Examples from Wells (can fetch from Google Books):

- A *polyhedron* is a solid bounded by polygons.
- The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.
- A frustum of any pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and for their bases the lower base, the upper base, and a mean proportional between the bases, of the frustum.

The Present

Definitions

Challenge: Look up definitions of *polygon* and *polyhedron* and create examples and nonexamples.

Euclid

Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

A *solid* is that which has length, breadth, and depth.

High School Text Example

A *polygon* is a closed figure in a plane, formed by connecting line segments endpoint to endpoint with each segment intersecting exactly two others.

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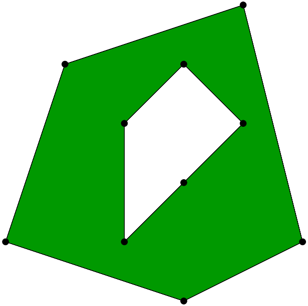
Annulus? Collinear sides?



High School Text Example

A solid formed by polygons that enclose a single region of space is called a *polyhedron*. . . . The flat polygonal surfaces of a polyhedron are called its *faces*. Although a face of a polyhedron includes the polygon and its interior region, we identify the face by naming the polygon that encloses it.

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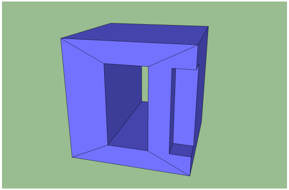

Polyhedral torus? Each edge shared by two polygons? Two polygons intersect in the empty set, single vertex, or single edge?

Example from College Text for High School Teachers

A *polygon* is a plane figure consisting of the points on n line segments (its sides), such that each side intersects exactly one other side at each of its endpoints (its vertices).

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Connected?

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Connected?

No formal definition for general polyhedron in this text.

Complexities of Definition of Polyhedron

- Lakatos, *Proofs and Refutations*.
- Grünbaum, Are Your Polyhedra the Same as My Polyhedra?, `http://www.math.washington.edu/~grunbaum/Your%20polyhedra-my%20polyhedra.pdf`

Present Common Presence of Polyhedra

- Definitions and examples of polyhedra and their parts
- Representation, visualization, and nets
- Classification of specific types of polyhedra (prisms, pyramids, etc.)
- Volume and surface area of specific types
- Platonic solids (When did this begin to be a common topic?)
- Euler's formula for polyhedra, $V - E + F = 2$ (When did this begin to be a common topic?)

NCTM and CCSSM

- National Council of Teachers of Mathematics *Principles and Standards for School Mathematics*.
  http://www.nctm.org/standards
- *Common Core State Standards for Mathematics* — new K-12 standards adopted by 44 states.
  http://www.corestandards.org/the-standards

"Polyhedron" does not appear as a term, though standard topics of volume and surface area do. Components, representation and visualization of three-dimensional objects are listed, but not Platonic solids or Euler's formula.

Opportunities

Challenges

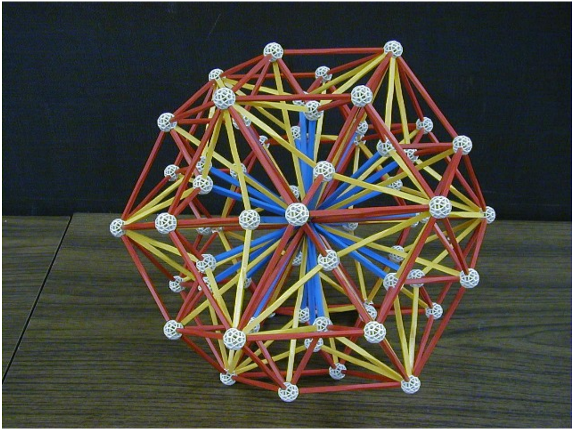
- Time needed for additional topics and high cognitive tasks
- Availability of advanced HS courses

Physical Models

Construction tools such as Polydron and Zome System are available (though expensive). Lots of models can be made cheaply with paper, cardboard, etc.

Image: http://www.mathlearningcenter.org/files/images/store/products/PDH.jpg

Physical Models



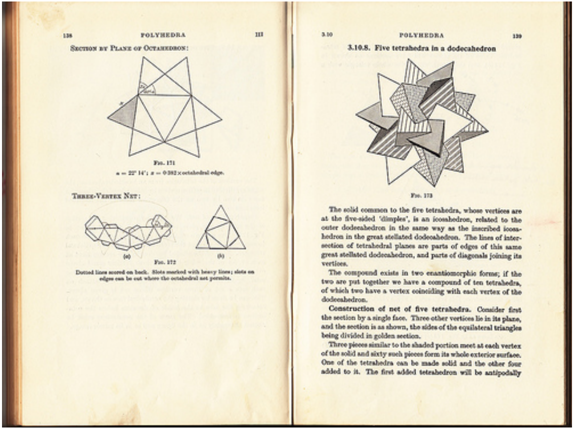
Physical Models

Hilbert, in *Geometry and the Imagination*:

"With the aid of visual imagination we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of the methods of investigation and proof, without necessarily entering into the details connected with the strict definitions of concepts and with the actual calculations."

Physical Models

Cundy and Rollett, in *Mathematical Models* (influenced me as a junior high school student):



138 POLYHEDRA 113

SECTION BY PLANE OF OCTAHEDRON:

FIG. 171

$\alpha = 22° 14'$; $x = 0{\cdot}382 \times$ octahedral edge.

THREE-VERTEX NET:

(a) (b)

FIG. 172

Dotted lines scored on back. Slots marked with heavy lines; slots on edges can be cut where the octahedral net permits.

3.10 POLYHEDRA 139

3.10.8. Five tetrahedra in a dodecahedron

FIG. 173

The solid common to the five tetrahedra, whose vertices are at the five-sided 'dimples', is an icosahedron, related to the outer dodecahedron in the same way as the inscribed icosahedron in the great stellated dodecahedron. The lines of intersection of tetrahedral planes are parts of edges of this same great stellated dodecahedron, and parts of diagonals joining its vertices.

The compound exists in two enantiomorphic forms; if the two are put together we have a compound of ten tetrahedra, of which two have a vertex coinciding with each vertex of the dodecahedron.

Construction of net of five tetrahedra. Consider first the section by a single face. Three other vertices lie in its plane, and the section is as shown, the sides of the equilateral triangles being divided in golden section.

Three pieces similar to the shaded portion meet at each vertex of the solid and sixty such pieces form its whole exterior surface. One of the tetrahedra can be made solid and the other four added to it. The first added tetrahedron will be antipodally

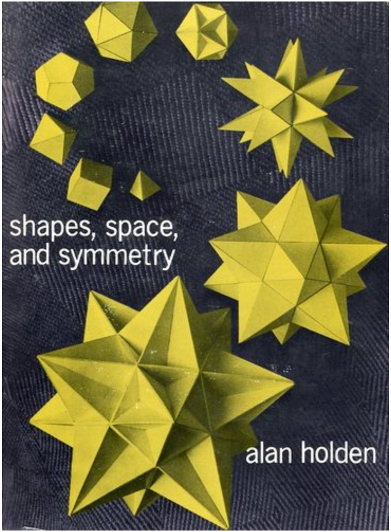
Image:

Physical Models

"Mathematics is often regarded as the bread and butter of science. If the butter is omitted, the result is indigestion, loss of appetite, or both. The purpose of this book is to suggest some ways of buttering the bread. The human mind can seldom accept completely abstract ideas; they must be derived from, or illustrated by, concrete examples. Here the reader will find ways of providing for himself tangible objects which will bring that necessary contact with reality into the symbolic world of mathematics."

Physical Models

Another influential book (personally):



Computer Geometry Visualization

One of the earliest examples of using computers to visualize geometrical objects: Banchoff movie—The Hypercube: Projections and Slicing.

http://www.math.brown.edu/~banchoff/Hypercube.html

Computer Geometry Visualization

Opportunity: Take advantage of free and powerful visualization software such as Google SketchUp, POV-Ray, and Blender (all currently used in some schools, but sometimes not in the mathematics classes). Consider coupling with 3D printers.

Computer Geometry Visualization

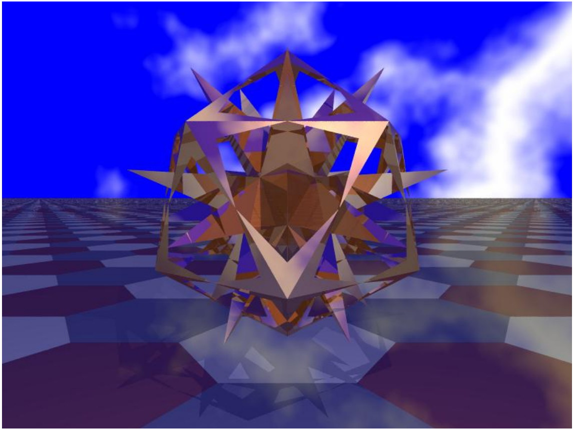


Image: Max Brown.

Computer Geometry Visualization

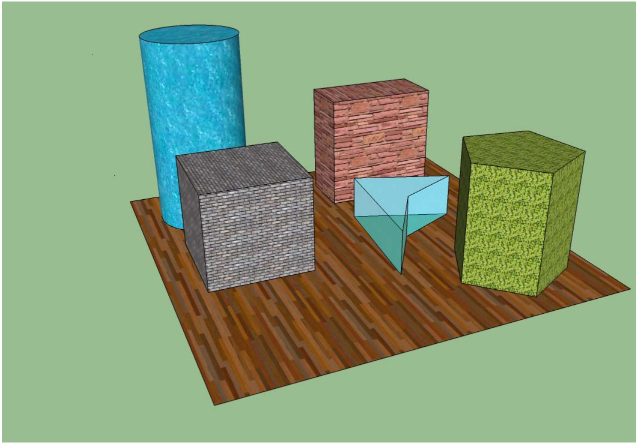


Image: Eighth Grade Student.

http://www.math.uky.edu/~lee/jessieclark/challenge1b.skp

Linear Programming

The graphical solutions of linear programs with two variables has appeared in high school algebra as an application of linear inequalities since the 1960's (at least).

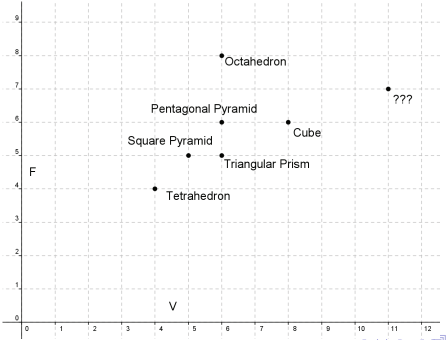
Opportunity: Move to three variables. Even just sketching the feasible regions might be helpful. Can couple this with the mathematics of visible and hidden face and edge recognition for a convex polyhedron.

Transformations

The CCSSM stresses the understanding of transformations.
Opportunity: Extend this into three-dimensions via an introduction to symmetry of polyhedra.

Euler's Formula

Opportunity: Consider some justifications of this formula. Investigate its domain of applicability. Develop Steinitz's necessary and sufficient conditions for the classification of (V, E, F) triples for convex polyhedra—a combination of constructions, elementary combinatorics and incidence, and algebra.



Regular and Semiregular Solids

Opportunity: After the Platonic solids, introduce the semiregular solids. Then study problems of symmetry, classification, and (V, E, F) calculations.

Representations

Opportunity: Some elementary investigations, even in two dimensions, between defining convex polygons as convex hulls and alternatively as solutions to sets of linear inequalities.

Crystals

Opportunity: Collaborate with science teachers to investigate crystal and packing problems.

Four Dimensions

Opportunity: Study some four dimensional polyhedra via their projections, Schlegel diagrams, and cross sections.

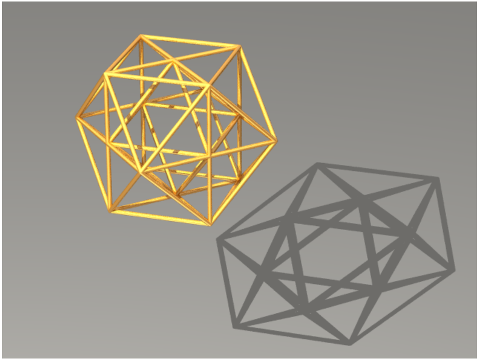


Image: Brian Vincent.

Opportunities

Other Ideas?