Transformations at the Heart of Connections and Creativity in STEAM

Carl Lee
University of Kentucky
www.ms.uky.edu/~lee/nctm2017/nctm2017.html

NCTM
April 2017
Reflection Cove #6 — 3:30–4:30 — next to room 214

Prelude

Pipedream (by Animusic)

`https://www.youtube.com/watch?v=hyCIpKAIFyo`

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- Sometimes relegated in K-12 math to a corner of geometry.
- Offer opportunities for strong connections to many concepts in math.
- Play significant roles in science, technology, engineering, and the arts.
- Lie at the heart of design software, including such free software as GeoGebra, SketchUp, POV-Ray, and Blender.

Plan for Today

Offer "snapshots" of problems and applications involving transformations.

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It is hoped that this will serve as a source of ideas that can be used, adapted, modified, and extended in various ways.

Some Free Software

- Desmos www.desmos.com
- GeoGebra www.geogebra.org
- SketchUp www.sketchup.com
- POV-Ray
  - PC www.povray.org
  - Mac megapov.inetart.net/povrayunofficial_mac
- Blender www.blender.org

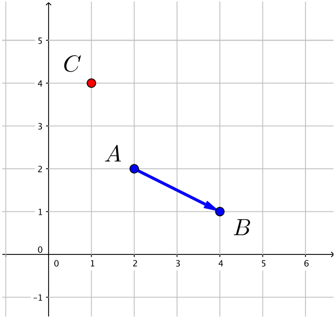
Rigid Motions

Rigid motions map the plane (or space) to itself without changing distances.

Translations

Translating a Point

Translate point C as indicated by vector $\overrightarrow{AB}$.



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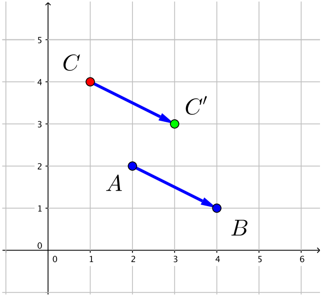
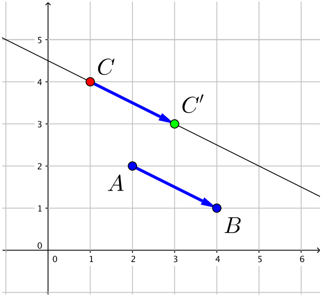


$$C' = C + (B - A).$$

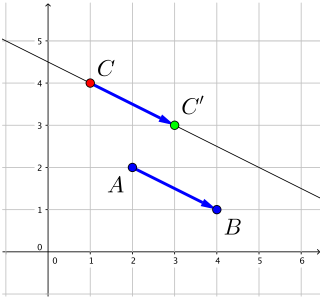
Lines



This connects to the parametric equation of a line $P(t) = C + t(B - A)$.

In this example, $P(t) = (1, 4) + t(2, -1)$.
$t = 1$ corresponds to the original translation.

Rectilinear Motion



This in turn connects to rectilinear motion — just vary t uniformly.

Make a slider for t in GeoGebra.
Type $(1, 4) + t * (2, -1)$ in the input space.
Turn on animation.
See `translatepoint4.ggb`.

Rectilinear Motion

Animation software can make this look fancier. I used POV-Ray to create the images and Blender to make the movie. The key command is

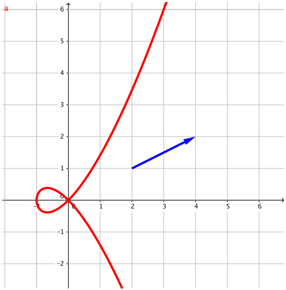
sphere$\{< 0, 1, 4 > +$clock$* < 0, 2, -1 >, 1$ texture$\{$T_Ruby_Glass$\}\}$.

(Note that we are looking directly at the x-axis towards the yz-plane.)

See `translatesphere.mov`.

Translating Curves

What if you want to translate a curve with a given equation?



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This connects to and explains the familiar "shifting" formulas that are seen in algebra, including shifting graphs of functions and writing equations of circles not centered at the origin.

Identifying Parabolas

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$$\overline{y} - k = 2(\overline{x} - h)^2 - 12(\overline{x} - h) + 23$$

or

$$\overline{y} = 2\overline{x}^2 + (-4h - 12)\overline{x} + (2h^2 + 12h + 23 + k).$$

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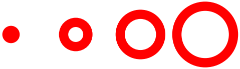
So the equation of the original parabola is $y - 5 = 2(x - 3)^2$ which has vertex $(3, 5)$.
We are using translation to "complete the square."

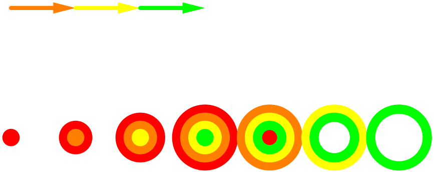
Using Translations to Make Patterns

Starting pattern



Using Translations to Make Patterns

Translate three times (images are in different colors to tell them apart)



Using Translations to Make Patterns

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This is the structure of the round "Row, Row, Row Your Boat" with four voices, twice through. (Translation in time.)

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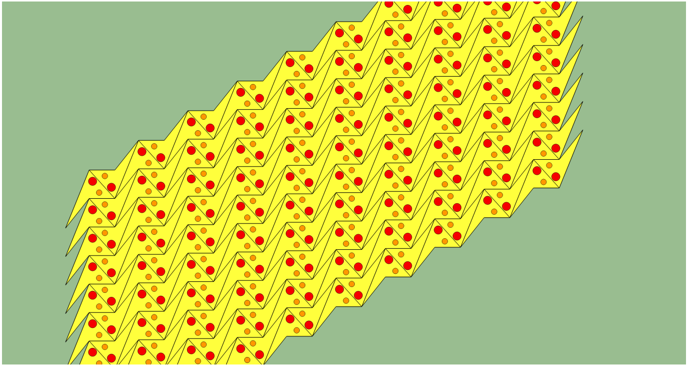
This is the structure of the round "Row, Row, Row Your Boat" with four voices, twice through. (Translation in time.)

What does translation in (the logarithm of the) pitch do?

Using Translations to Make Patterns



Tiling by quadrilaterals made with SketchUp. Adjacent quadrilateral is obtained by rotation about midpoint of shared edge. Then pairs of quadrilaterals are translated.

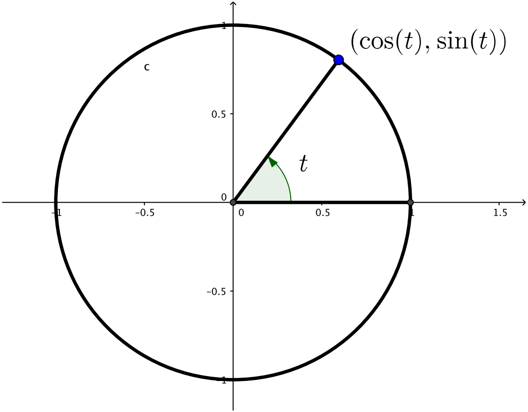
Translations in Time

For another entertaining example, see the video of Kylie Minogue's "Come into my World,"
`https://www.youtube.com/watch?v=63vqob-MljQ`.

Rotations

The Power of Trig

Consider a point on the unit circle



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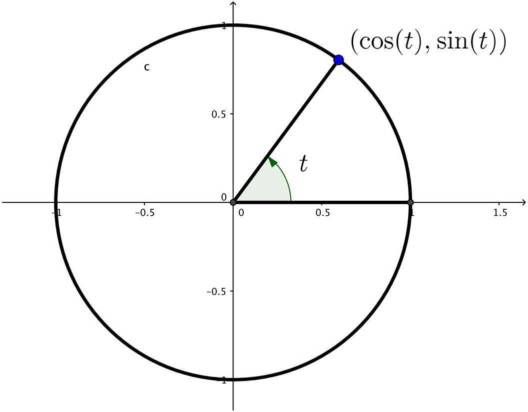


As t increases, the point rotates counterclockwise about the origin along the path of the circle.

The Power of Trig

Try this in GeoGebra.

Make a slider for t, selecting the "angle" option.

Type $(\cos(t), \sin(t))$ in the input space.

Turn on animation.

See `unitcircle2.ggb`.

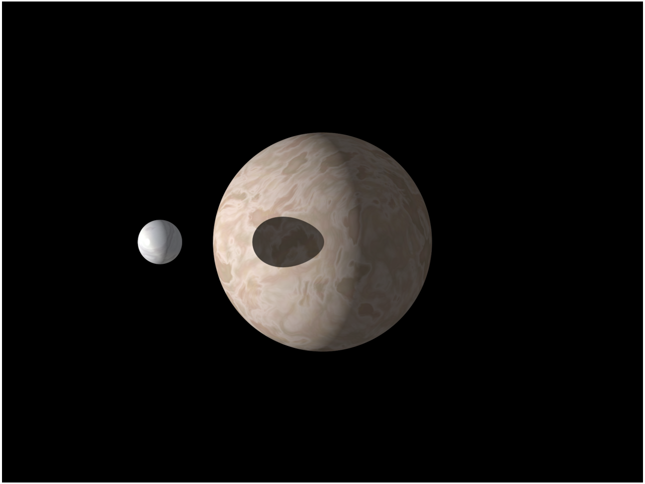
Simple Planetary Motion

Suppose a planet rotates counterclockwise around its axis three times while at the same time it revolves once counterclockwise around the sun. How many days do the inhabitants experience during the year?

See `planet.ggb`.

Simple Planetary Motion

A fancier planet and moon created with POV-Ray and Blender.



See `rotatesphere.mov`.

Motions with Logo "Turtle Graphics"

Logo is a language to construct drawn figures using motion commands for moving and turning, etc.

See `https://turtleacademy.com` and `turtle.mov`, an example from this website.

Adding and Multiplying Vectors

Add vectors in the usual way, placing them tail to head.
See `vectorsum.ggb`.
This is essentially translation.

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Multiply vectors by multiplying their lengths and adding their angles.
See `vectorproduct.ggb`.
This is rotation and dilation.

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Find a vector A such that $A^3 = 1$. Find another.

Complex Numbers

We have just seen the geometric model for the complex numbers. "Under the hood" are the trig angle sum identities, which lead directly to the important rotation formula:

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

to rotate a point (x, y) counterclockwise about the origin by an angle t.

Rotating Conics

Problem from my precalculus high school course:
What is the resulting equation if the parabola described by $y = x^2$ is rotated counterclockwise about the origin by the angle t having $\sin t = \frac{7}{25}$ and $\cos t = \frac{24}{25}$?

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We use the rotation formula:

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Substituting, we have

$$-\frac{7}{25}\overline{x} + \frac{24}{25}\overline{y} = \left(\frac{24}{25}\overline{x} + \frac{7}{25}\overline{y}\right)^2$$

which simplifies to

$$576\overline{x}^2 + 336\overline{xy} + 49\overline{y}^2 + 175\overline{x} - 600\overline{y} = 0.$$

Rotating Conics

Another problem from the same course:
Analyze the conic given by the equation

$$73x^2 - 72xy + 52y^2 - 410x + 120y + 525 = 0.$$

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We wish to apply a rotation by angle t that eliminates the xy term. We use the rotation formulas. Let's abbreviate $s = \sin t$ and $c = \cos t$.

$$\begin{aligned} x &= c\bar{x} + s\bar{y}, \\ y &= -s\bar{x} + c\bar{y}. \end{aligned}$$

After substitution and simplification we find that the coefficient of $\overline{xy}$ is

$$42sc - 72(c^2 - s^2).$$

Rotating Conics

We need an angle t so that this expression equals 0. Let $T = 2t$, $S = \sin T$, and $C = \cos T$. Then $S = 2sc$ and $C = c^2 - s^2$ by the Double Angle Formulas. So we want an angle T with

$$21S - 72C = 0.$$

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$$21S - 72C = 0.$$

But this means $\tan T = \frac{S}{C} = \frac{24}{7}$. From this (and the Pythagorean Theorem) we calculate $S = \frac{24}{25}$ and $C = \frac{7}{25}$.

Rotating Conics

Now we use the Half Angle Formulas to find s and c:

$$s = \sqrt{\frac{1-C}{2}} = \frac{3}{5},$$

$$c = \sqrt{\frac{1+C}{2}} = \frac{4}{5}.$$

Using these values of c and s, the rotated conic has equation

$$100\overline{x}^2 - 400\overline{x} + 25\overline{y}^2 - 150\overline{y} + 525 = 0.$$

Rotating Conics

Complete the two squares to get

$$100(\overline{x}^2 - 4\overline{x} + 4) + 25(\overline{y}^2 - 6\overline{y} + 9) = 100,$$

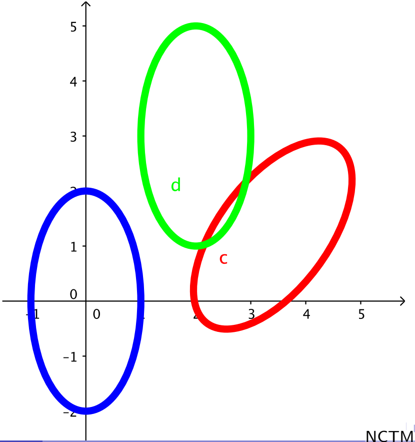
or

$$(\overline{x} - 2)^2 + \frac{(\overline{y} - 3)^2}{4} = 1.$$

This is an ellipse with center $(2, 3)$.

Rotating Conics

So we have deduced that the original ellipse can be obtained from the ellipse $\overline{x}^2 + \frac{\overline{y}^2}{4} = 1$ by first translating it by $(2, 3)$ and then rotating it clockwise by the angle t with $\sin t = \frac{3}{5}$ and $\cos t = \frac{4}{5}$.



Fitting Functions to Figures

In Desmos type

$f(x) = \sin(x)$

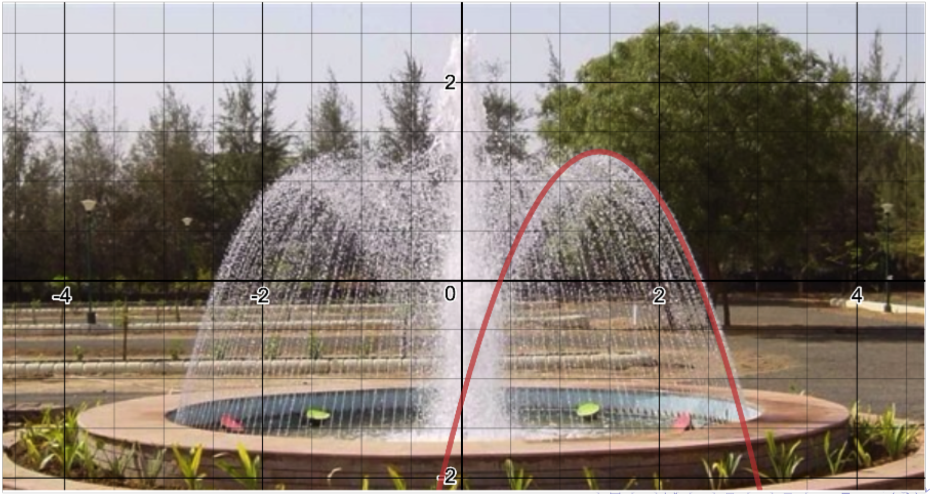
(or your favorite function!), and

$g(x) = a * f((x - c)/b) + d$.

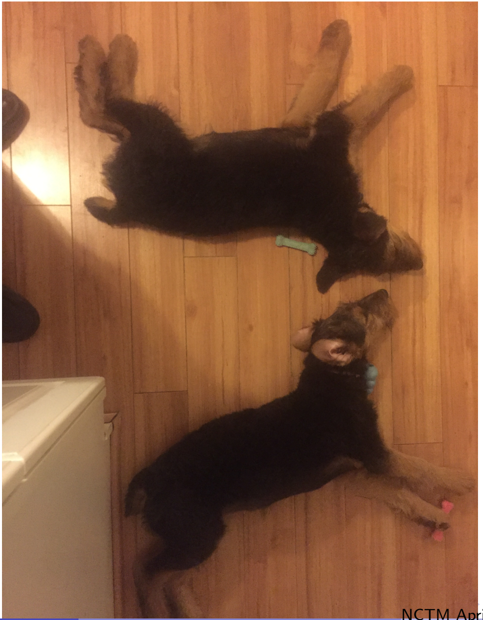
Accept the offer to create four sliders for a, b, c, d and experiment with changing their values.

Fitting Functions to Figures

Now insert a photo into Desmos and, changing $f(x)$ to an appropriate function, try to match an element of the photo with the graph.



Reflections



Reflections Real and Mathematical

Why does the apparent location of a reflected object match the defined location of the mathematical reflection of that object?

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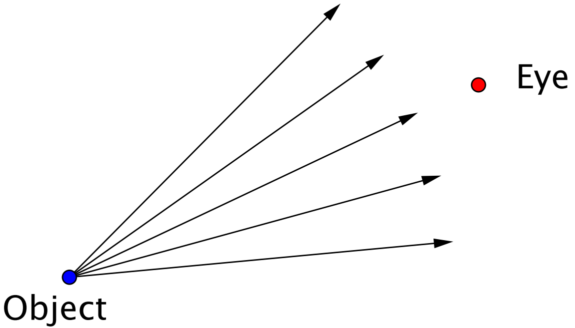
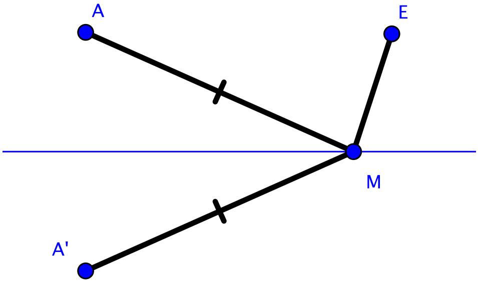
First Key Idea: The brain perceives the location of an object to be at the confluence of rays of light coming from that location.



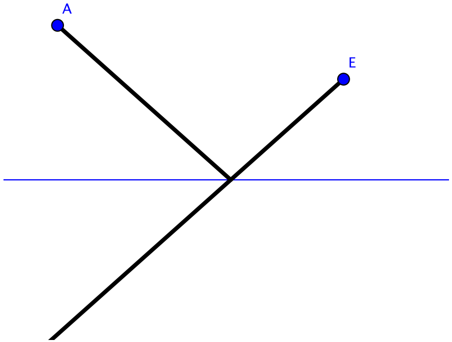
Reflections Real and Mathematical

Second Key Idea: Light travels the path of least time, and this implies the angle of incidence equals the angle of reflection. See `vision2.ggb`—move the point M.

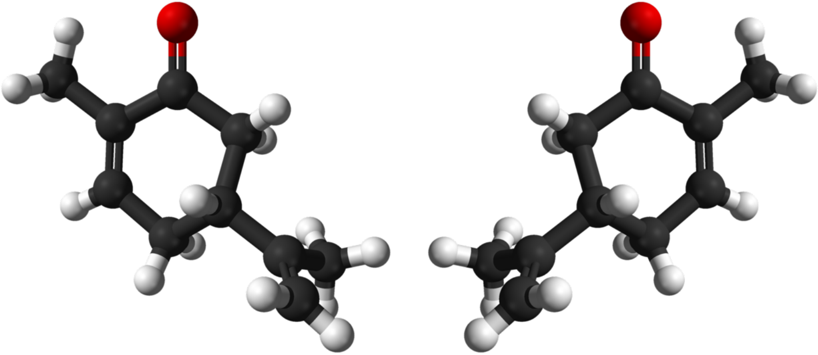


Reflections Real and Mathematical

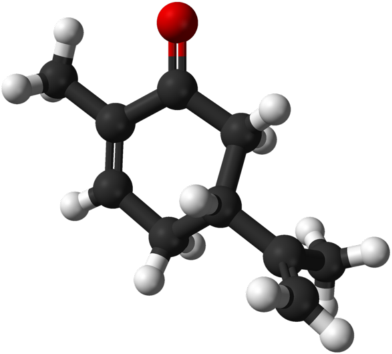
Now move the point E in `vision3.ggb` to see that the reflected rays reaching E appear to trace back and converge on the mathematical reflection of A in the mirror.



Reflections in Chemistry



Reflections in Chemistry



Spearmint and Caraway
(R-carvone and S-carvone)
Reflecting a compound may dramatically change its properties.

Anamorphic Art

Look at this distorted image as it is reflected in a mirrored cylinder.

Anamorphic Art

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See `anamorph2.mov`, made with POV-Ray and Blender.

For Later Discussion?

Why does a mirror reverse an image left and right, but not up and down?

Glide Reflections

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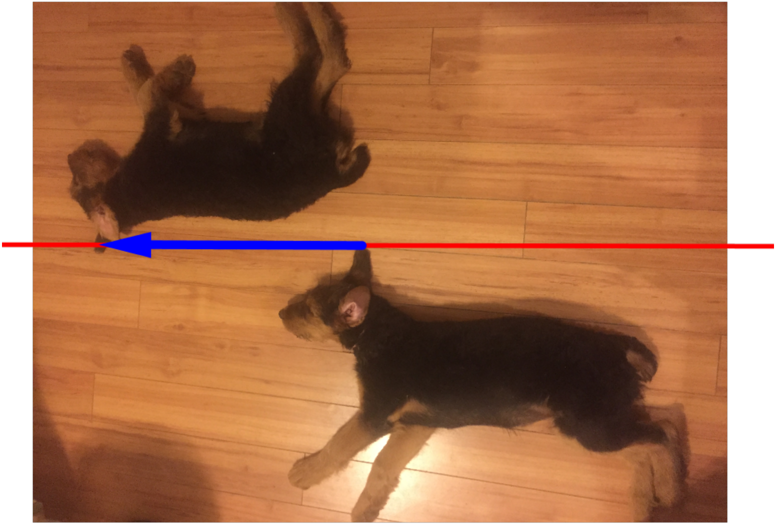

Classification

Every rigid motion of the plane is one of the following: a translation, a rotation, a reflection, or a glide reflection.

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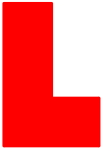
Throw two copies of the following shape on the floor and identify the rigid motion mapping one to the other.



Questions to Ask with Technology

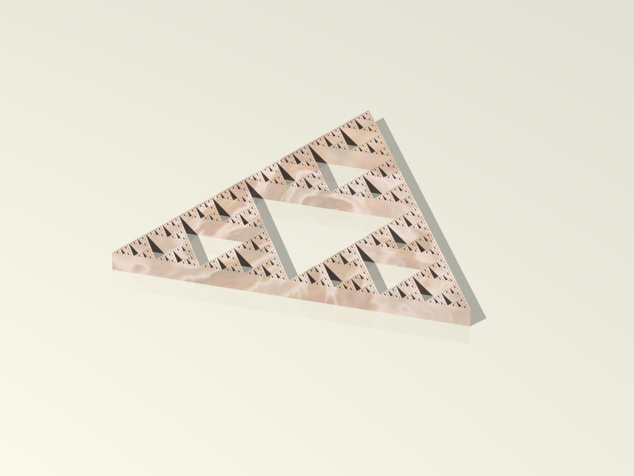
For each of the following files, precisely determine what the rigid motion is. (Move the point A.)

`iso50.ggb`
`iso60.ggb`
`iso70.ggb`
`iso80.ggb`

Dilations

Fractals

Sierpinski Triangle



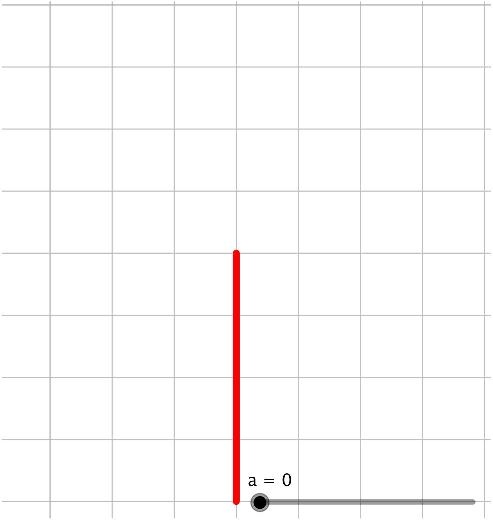
See `sierpinski2.mov`, made with POV-Ray.

Fractals

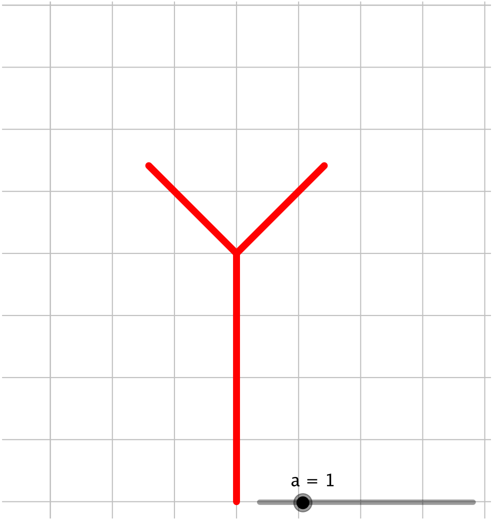
Start with a simple element; dilate, replicate, reposition, to make a new figure; repeat with the new figure.

See the file `fractal3.ggb`.

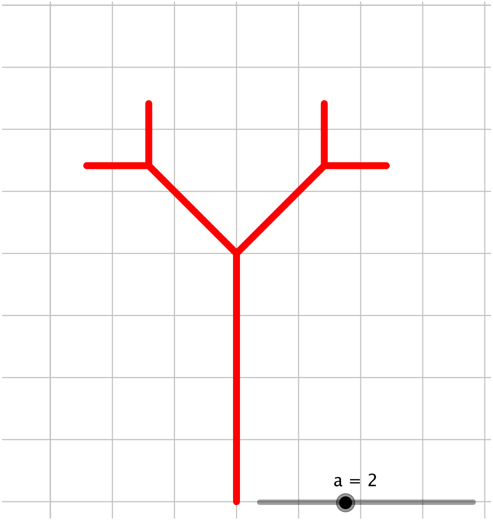
Fractals



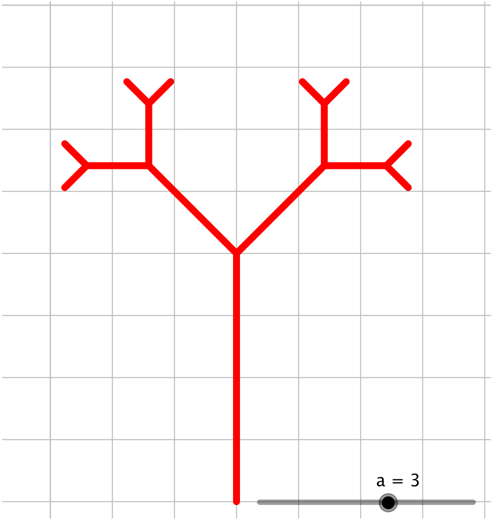
Fractals



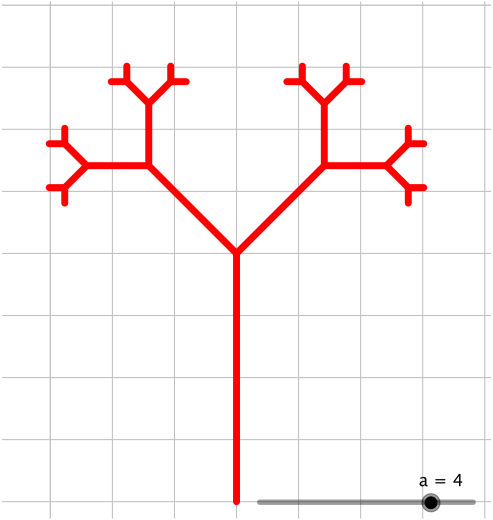
Fractals



Fractals



Fractals



Fractals

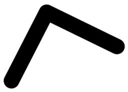
A fancier version made with POV-Ray.

Dragon Curve

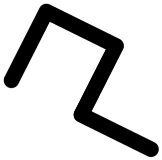
A fractal made with rotations.

Fold a long strip of paper in half from, say, right to left. Then fold it again from right to left. And again. And again. . . . Then unfold the entire strip until each crease forms a right angle.

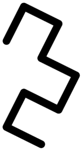
Dragon Curve



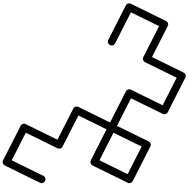
Dragon Curve



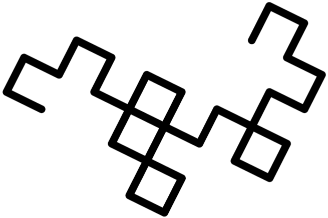
Dragon Curve



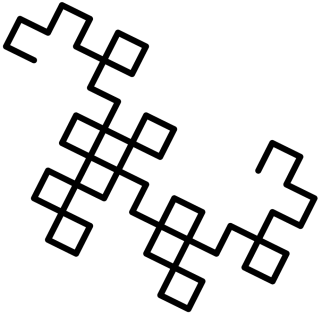
Dragon Curve



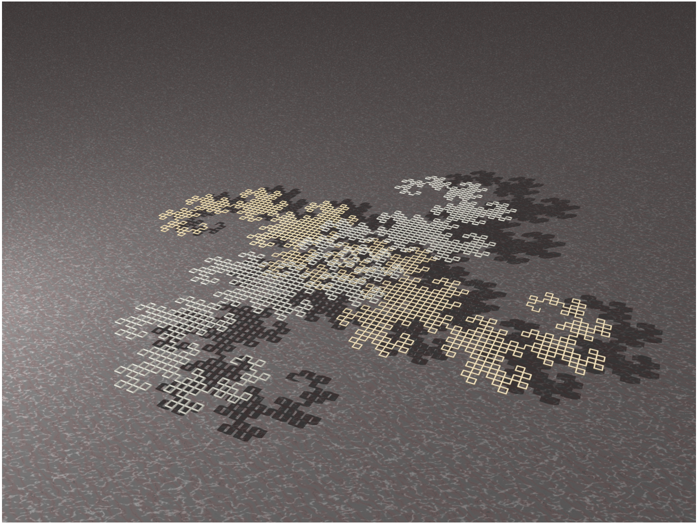
Dragon Curve



Dragon Curve



Dragon Curve

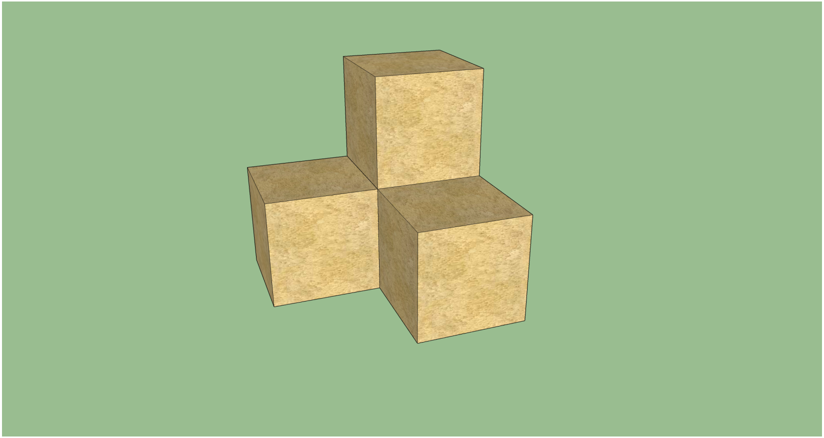


Classifying Molecules

Molecules can be classified according to their symmetries—what sets of 3D rigid motions leave the molecules unchanged in appearance. See, for example,
`https://en.wikipedia.org/wiki/Molecular_symmetry`.

Classifying Molecules

Make your own "molecules" from cubes and study their rotational and reflectional symmetries. How many molecules can you make with different sets of symmetries? How many ways can you draw a given molecule on isometric dot paper?



Square Dancing

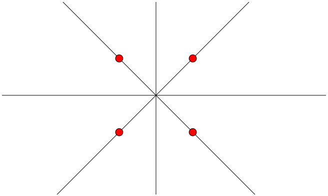
What transformations do the women represent?
What transformations do the men represent?

`ladies2.mov`
`ladies4.mov`

Animations from `http://www.squaredancecd.com/sdance.htm`.

Square Dancing

You can do simpler versions with four individuals in your class.



The dance "calls" could be

- Rotate 90, 180, or 270 degrees clockwise.
- Reflect across the x-axis, y-axis, the line $y = x$, or the line $y = -x$.

(You can think of more creative names!)

Consider the net result of a sequence of moves. Find a single move that can "undo" a sequence of moves.

Art — iOrnament

Powerful iPad app using transformations and symmetry systems to create beautiful images.

https://itunes.apple.com/us/app/iornament-draw-creative-geometry/id534529876?mt=8

Three-D Design — Jessie Clark Middle School

Programs like SketchUp and Blender are fundamentally based on transformations—some very sophisticated and powerful. Here is an example from an eighth grader at Jessie Clark Middle School.

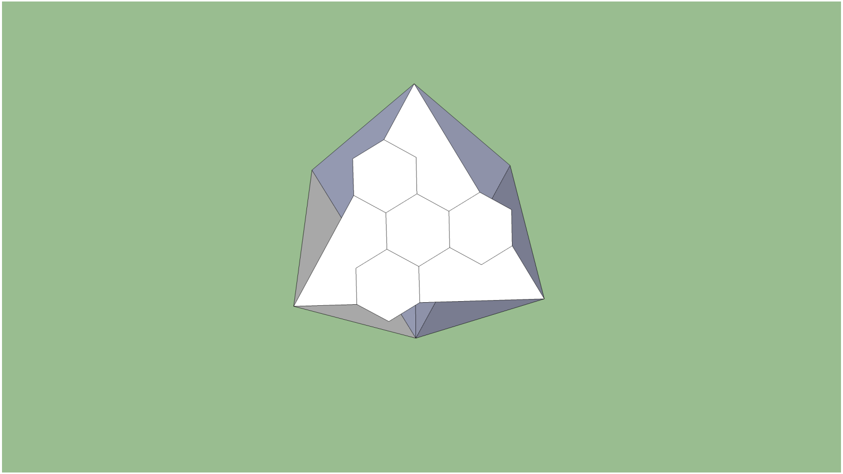
Here is the SketchUp file: `room1.skp`.

See also a summary of SketchUp and transformations, `SketchUp.pdf`.

Three-D Design — Jessie Clark Middle School

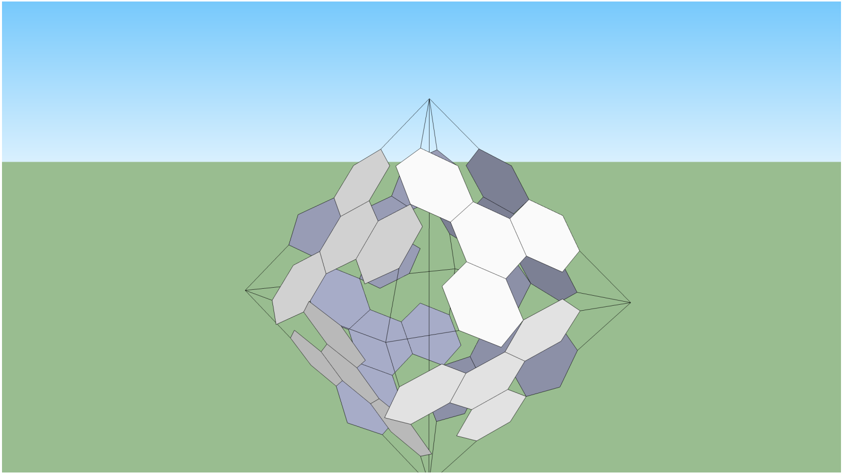
Three-D Design and Printing

An example of a puzzle constructed in SketchUp



Three-D Design and Printing

An example of a puzzle constructed in SketchUp



Three-D Design and Printing

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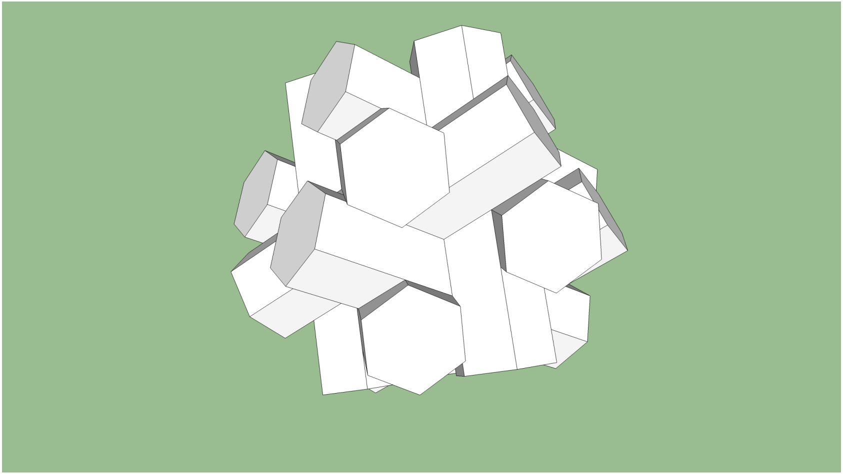

Three-D Design and Printing

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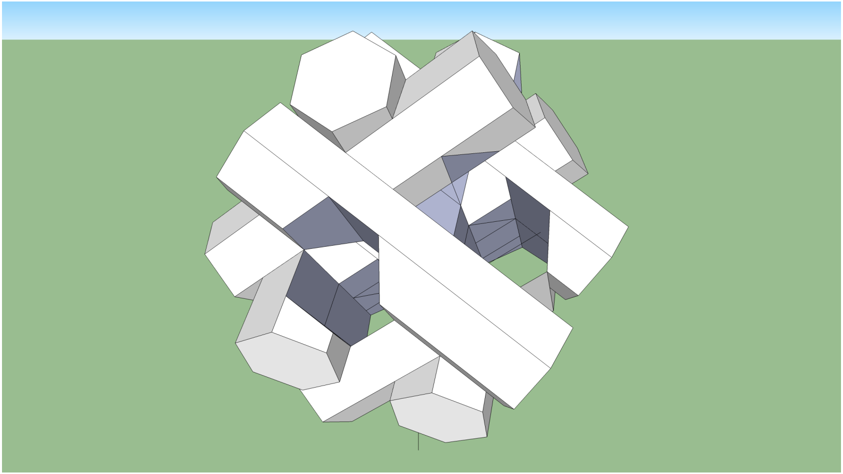

Three-D Design and Printing

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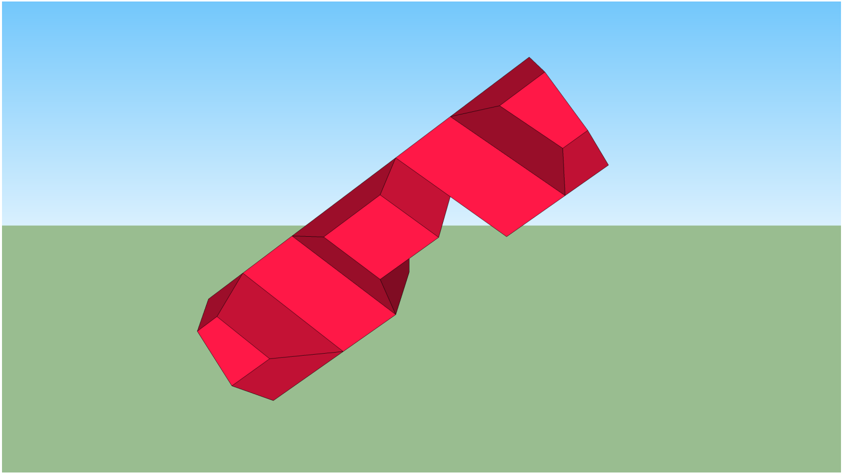

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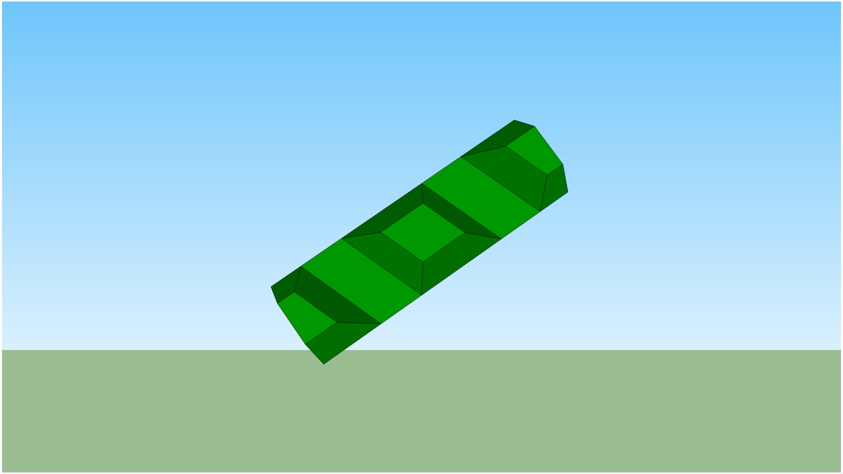

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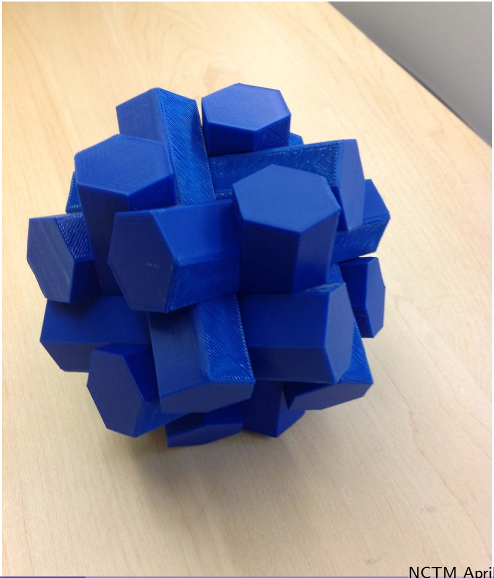
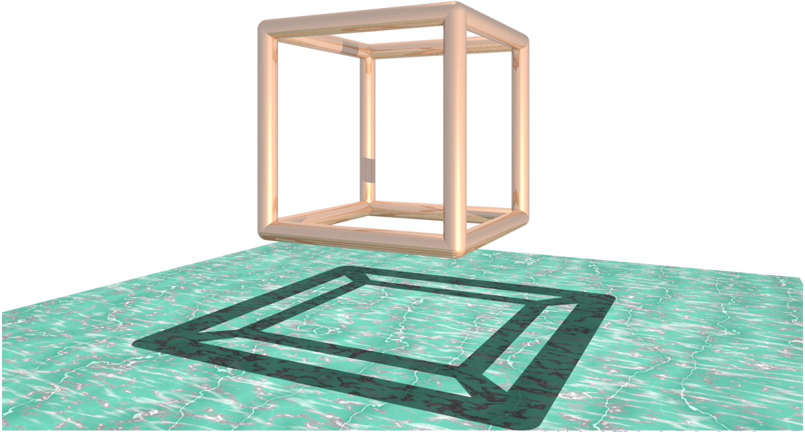


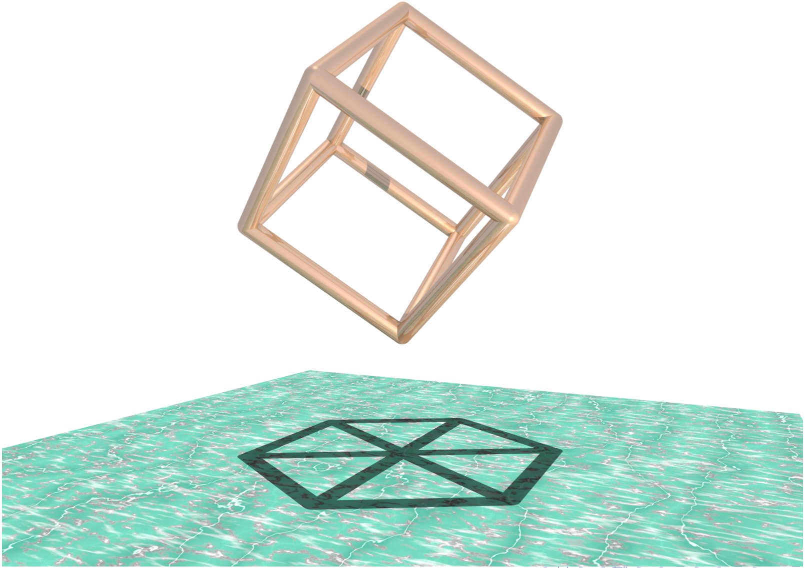
Projections

Omnipresent—every time we look at a three-dimensional object as displayed on a two-dimensional surface or screen

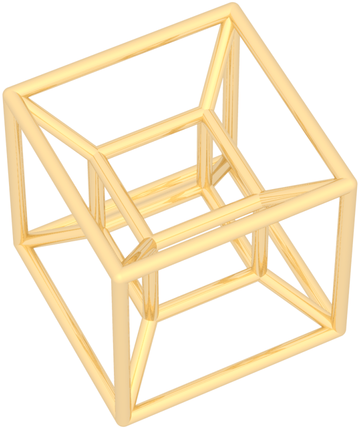
Projection of a Three-Dimensional Cube



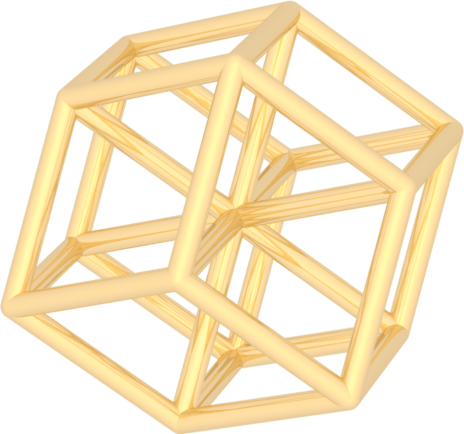
Projection of a Three-Dimensional Cube



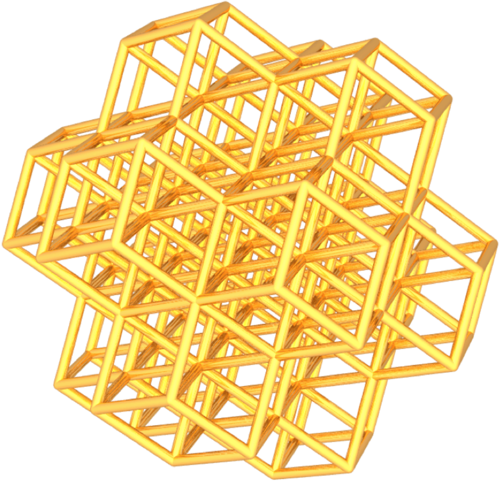
Projection of a Four-Dimensional Cube



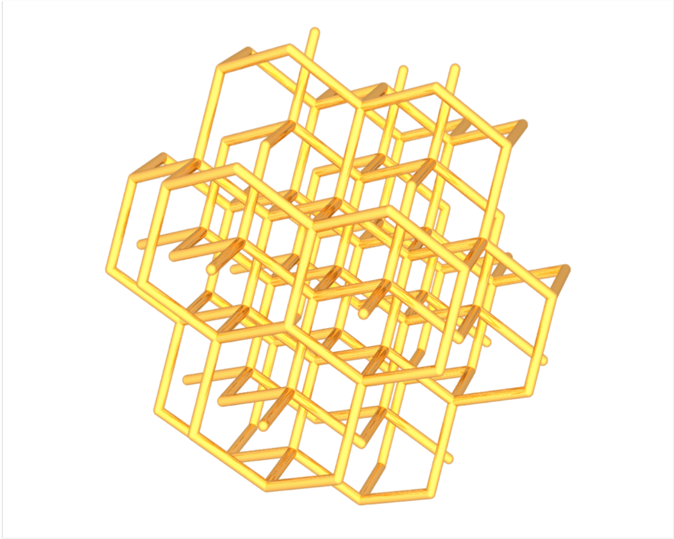
Another Projection of a Four-Dimensional Cube



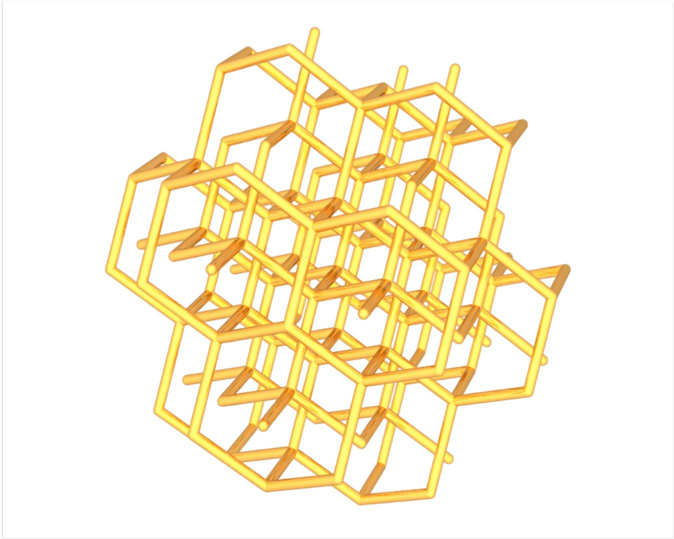
Tile by Translation



Remove Bars Strategically



Remove Bars Strategically—Diamond Crystal!



Parting Thoughts

How can we make more of these rich, reinforcing connections among math, science, technology, engineering, and art in the classroom?

Parting Thoughts

How can we make more of these rich, reinforcing connections among math, science, technology, engineering, and art in the classroom?

P.S. I learned about POV-Ray from a high school student, so I recommend letting your students learn to use these programs and then show you!

Thank you!

See http://www.ms.uky.edu/lee/nctm2017/nctm2017.html for this talk and related documents and media. The talk is also available through the NCTM conference app.

Reflection Cove #6 — 3:30–4:30 — next to room 214

Images

- Fountain: http://weknowyourdreams.com/image.php?pic=/images/fountain/fountain-04.jpg
- Carvone: https://en.wikipedia.org/wiki/Carvone
- Anamorphic Art: https://annonetheelephant.com/2012/04/17/anamorphic-annone
- iOrnament: http://www.spektrum.de/alias/dachzeile/ornament-wettbewerb/1223589