The Many Facets of Polyhedra

Carl Lee
University of Kentucky

Ohio Section of the MAA — March 2015

Let's Build!

Using a construction set like Polydron, construct some physical closed "solids".

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What did you make?

Inspiration

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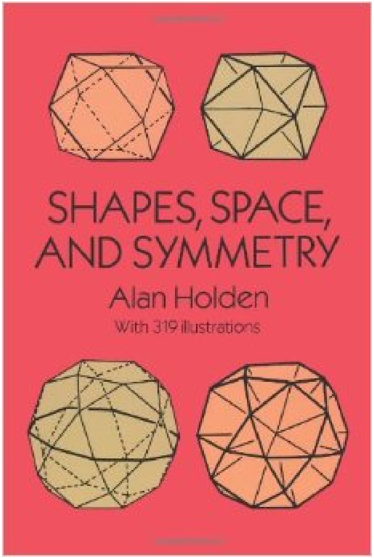
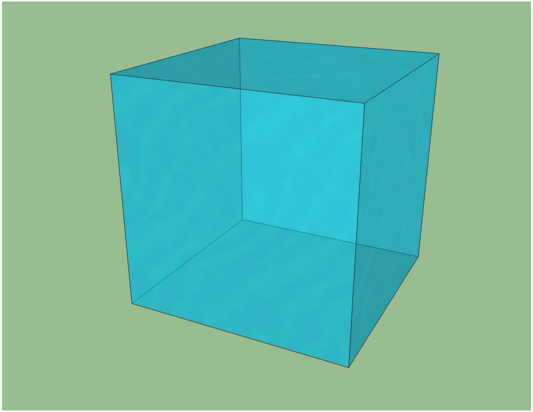


Construction

Polyhedra

A convex polytope P is the convex hull of (smallest convex set containing) a finite set of points in $\mathbf{R}^d$.

Example: Cube



Nets

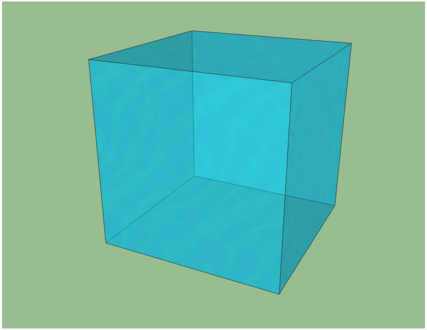
We commonly make non-overlapping nets for three-dimensional polytopes (cutting along the edges) but it has not been proven that this is possible for every polytope.

However, if we allow cutting across faces, then it is always possible to make a non-overlapping net.

Wonderful video by Eric Demaine to show the complexity of such a “simple” idea:
`http://erikdemaine.org/metamorphosis`

Virtual Constructions

SketchUp examples
`http://www.sketchup.com` (Free!)



Cube

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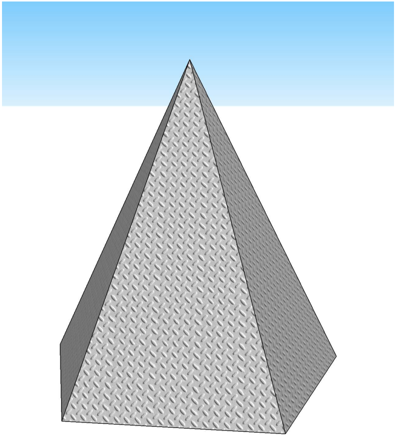

Pyramid

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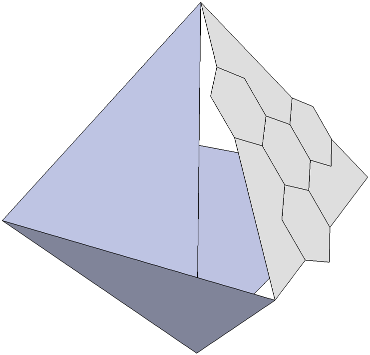

Polyhedral Puzzle

Virtual Constructions

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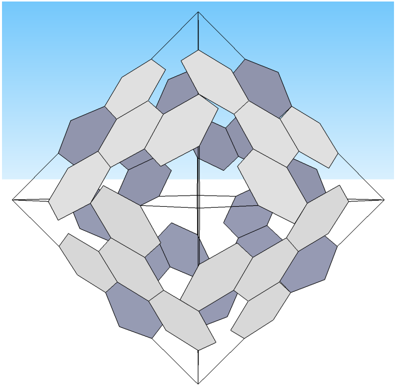

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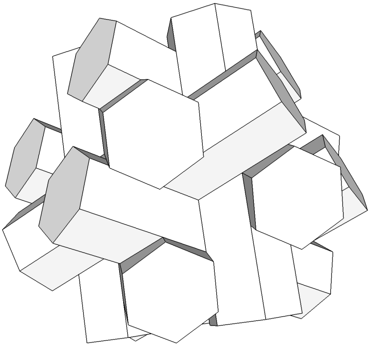

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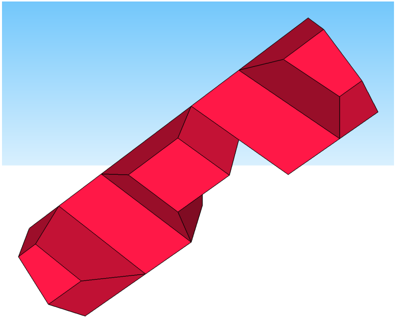

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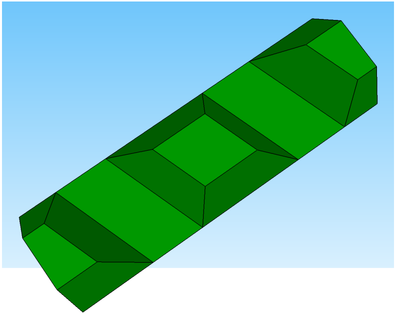
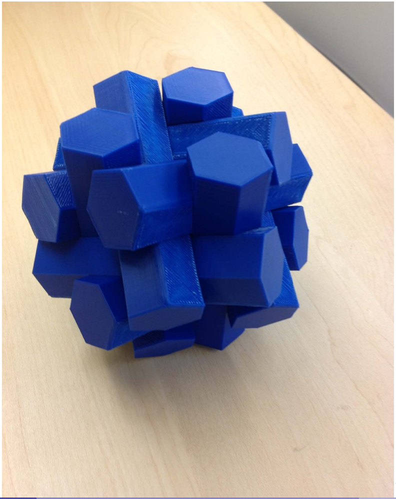


Polyhedral Puzzle

3D Printed Constructions

Now just send the files to the 3D printer!
For example, export to Makerbot Desktop,
`http://www.makerbot.com/desktop` (Also free!)



Other Virtual and 3D Printing Construction Software

- Tinkercad, https://www.tinkercad.com
- 123D Catch, http://www.123dapp.com/catch
- 123D Design, http://www.123dapp.com/design
- OpenSCAD, http://www.openscad.org
- Blender, http://www.blender.org
- POV-Ray, http://www.povray.org

Properties

Back to Polytopes!

Two ways to describe polytopes:

- By their vertices
- By their defining inequalities

Two Descriptions

The cube is the convex hull of its vertices

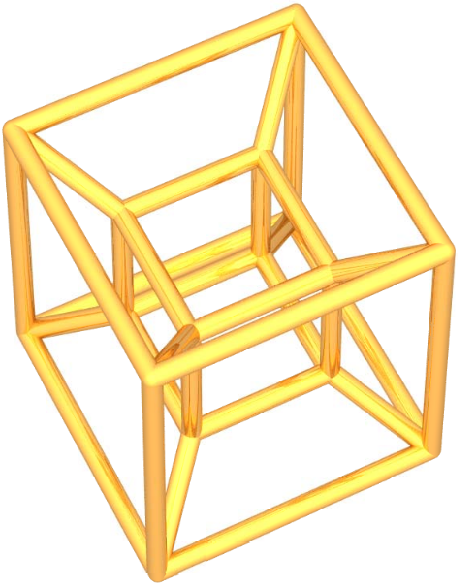
$$\begin{array}{c}(0,0,0)\\(0,0,1)\\(0,1,0)\\(0,1,1)\\(1,0,0)\\(1,0,1)\\(1,1,0)\\(1,1,1)\end{array}$$

Two Descriptions

The cube is defined by the inequalities

$$
\begin{aligned}
x &\geq 0 \\
x &\leq 1 \\
y &\geq 0 \\
y &\leq 1 \\
z &\geq 0 \\
z &\leq 1
\end{aligned}
$$

Hypercubes—A Peek into the Fourth Dimension



Hypercubes

The hypercube (4-dimensional cube) is the convex hull of its vertices

$$
\begin{array}{c}
(0,0,0,0)\\
(0,0,0,1)\\
(0,0,1,0)\\
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Hypercubes

The hypercube is defined by the inequalities

$$\begin{aligned} x_1 &\geq 0 \\ x_1 &\leq 1 \\ x_2 &\geq 0 \\ x_2 &\leq 1 \\ x_3 &\geq 0 \\ x_3 &\leq 1 \\ x_4 &\geq 0 \\ x_4 &\leq 1 \end{aligned}$$

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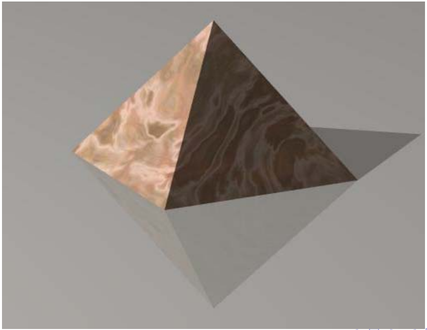
(The algebra may seem prosaic but the object is nevertheless entrancing.)

Two Descriptions

This result lends itself to practical considerations when constructing polytopes.

For example, POV-Ray can use either description. See
`http://www.ms.uky.edu/~lee/visual05/povray/pyramid1.pov`
and
`http://www.ms.uky.edu/~lee/visual05/povray/pyramid2.pov`
each of which constructs this image:



Two Descriptions

But this suggests an important computational question: How can you convert from one description to the other?
This is an example of a question in computational geometry.

Possibilities

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  (Why even?—Let's shake hands on it.)

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  SketchUp can help here!

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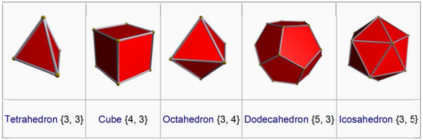
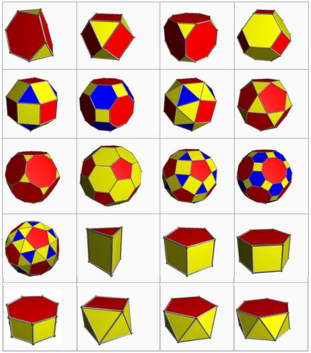
What Three-Dimensional Polytopes Can We Make?

Platonic (Regular) Convex Polyhedra



Semiregular Convex Polyhedra



(Images from Wikipedia)

Limitations

Counting Vertices, Edges, and Faces

Where geometry meets discrete math.

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Let's count the number of vertices, V, edges, E, and faces, F, of a three-dimensional polytope, and write as a list (V, E, F), called the face-vector.

We can extend this idea to counting the elements of higher dimensional polytopes as well.

Counting Vertices, Edges, and Faces

Example:

- Cube. $(8, 12, 6)$.

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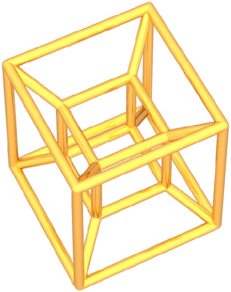
- Hypercube. $(16, 32, 24, 8)$.



Counting Vertices, Edges, and Faces

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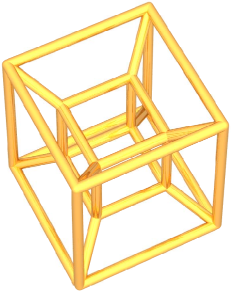

Question: What are the possible face-vectors of polytopes?

Three-Dimensional Polytopes

Theorem (Euler's Relation)

$V - E + F = 2$ for convex 3-polytopes.

Example: Cube. $8 - 12 + 6 = 2$.

Three-Dimensional Polytopes

Sketch of proof: Sweep the polytope with a plane in general direction. (Think of immersing in water.) Count vertices, edges, and polygons only when fully swept (under water). Watch how $\chi = V - E + F$ changes when the plane hits each vertex.

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Total change in χ is therefore 2.

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Note: This proof technique generalizes to higher dimensions.

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V, E, F are positive integers.

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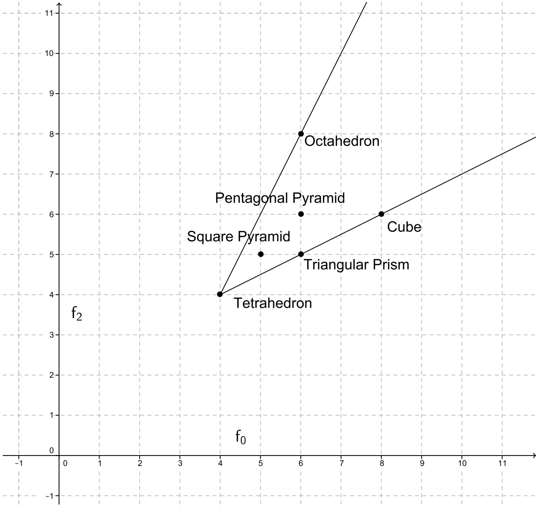
Theorem (Steinitz)

A positive integer vector (V, E, F) is the face-vector of a 3-polytope if and only if the following conditions hold.

- $V - E + F = 2$,
- $V \leq 2F - 4$, *and*
- $F \leq 2V - 4$.

The World of Three-Dimensional Polytopes

(Here, V is labeled f_0 and F is labeled f_2.) Think about how to construct representatives of each face-vector!



Four-Dimensional Polytopes

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We don't know!—It's an area of active research
But there are some partial results.

The Amazing Power of Euler

Steinitz's Inequalities

Let's do a little more math.

Let F_i be the number of faces with i edges.

Let V_i be the number of vertices incident to i edges.

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$$\begin{array}{rcll} 2E & = & 3F_3 + 4F_4 + 5F_5 + \cdots & \text{Take it apart!} \\ & \geq & 3F_3 + 3F_4 + 3F_5 + \cdots & \end{array}$$

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$$2E = 3V_3 + 4V_4 + 5V_5 + \cdots$$

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$$2V + 2F - 4 = 2E \geq 3F$$

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$$2V + 2F - 4 = 2E \geq 3F$$

$$2V - 4 \geq F$$

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$$2V + 2F - 4 = 2E \geq 3F$$
$$2V - 4 \geq F$$

$$2V + 2F - 4 = 2E \geq 3V$$
$$2F - 4 \geq V$$

But Wait! There's More!

$$6 = 3V - 3E + 3F$$

But Wait! There's More!

$$
\begin{aligned}
6 &= 3V - 3E + 3F \\
&\leq 3V - 3E + 2E
\end{aligned}
$$

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$$\begin{aligned} 6 &= 3V - 3E + 3F \\ &\leq 3V - 3E + 2E \\ 6 &\leq 3V - E \end{aligned}$$

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$$12 \leq 6F - 2E$$

But Wait! There's More!

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12 &\leq 6F - 2E \\
&= 6(F_3 + F_4 + F_5 + \cdots) - (3F_3 + 4F_4 + 5F_5 + \cdots)
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Theorem

Every polytope must have at least one triangle, quadrilateral, or pentagon as a face.

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Theorem

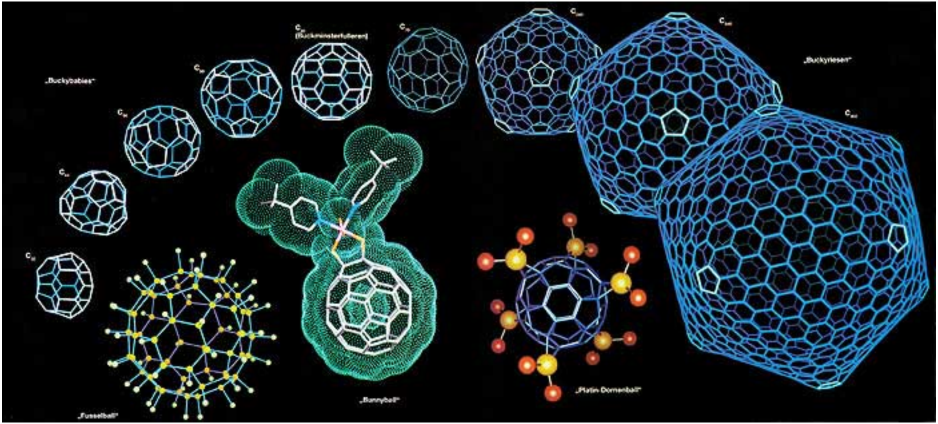
Every polytope must have at least one triangle, quadrilateral, or pentagon as a face.

In a similar way

Theorem

Every polytope must have at least one vertex of degree 3, 4, or 5.

Fullerenes



Carbon compounds forming spheres of pentagons and hexagons with every vertex of degree 3.

Fullerenes

$$3V = 2E = 5F_5 + 6F_6 = 6F - F_5$$

Fullerenes

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$$6V - 6E + 6F = 12$$

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Fullerenes

$$3V = 2E = 5F_5 + 6F_6 = 6F - F_5$$

$$6V - 6E + 6F = 12$$

$$4E - 6E + 2E + F_5 = 12$$

$$F_5 = 12$$

Fullerenes

$$3V = 2E = 5F_5 + 6F_6 = 6F - F_5$$

$$6V - 6E + 6F = 12$$

$$4E - 6E + 2E + F_5 = 12$$

$$F_5 = 12$$

Theorem

Every fullerene must have exactly 12 pentagons.

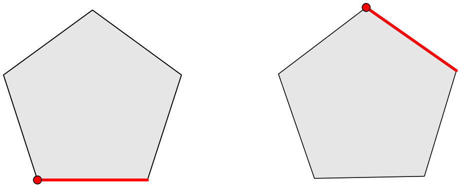
Symmetry

Regular Polygons

What about its symmetries makes a polygon regular?

Regular Polygons

What about its symmetries makes a polygon regular?



Choose any two vertex-edge pairs (v, e) and (v', e') such that v is an endpoint of e and v' is an endpoint of e'.

Then there is a symmetry of the polygon that maps (v, e) to (v', e').

That is to say, the symmetry group of a regular polygon is flag-transitive.

Regular Polygons

Can you think of some polygons whose symmetry groups are

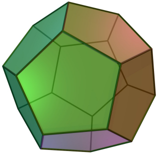
- Vertex transitive but not edge transitive?
- Edge transitive but not vertex transitive?

Platonic Solids

What about its symmetries characterizes a Platonic solid?

Platonic Solids

What about its symmetries characterizes a Platonic solid?



Choose any two vertex-edge-face triples (v, e, f) and (v', e', f') such that v is an endpoint of e and e is an edge of f, and also v' is an endpoint of e' and e' is an edge of f'.
Then there is a symmetry of the polytope that maps (v, e, f) to (v', e', f').
That is to say, the symmetry group of a Platonic solid is flag-transitive.

Regular Polytopes in Higher Dimensions

This notion of regularity extends naturally into higher dimensions.

- In four dimensions there are 6 regular polytopes.
- In five and higher dimensions there are only 3.

Regular Polytopes

Can you think of some three-dimensional polytopes whose symmetry groups are

- Vertex transitive only?
- Edge transitive only?
- Face transitive only?
- Vertex-edge transitive?
- Edge-face transitive?
- Vertex-face transitive?

Semiregular Solids

What about its symmetries characterizes a semiregular solid?

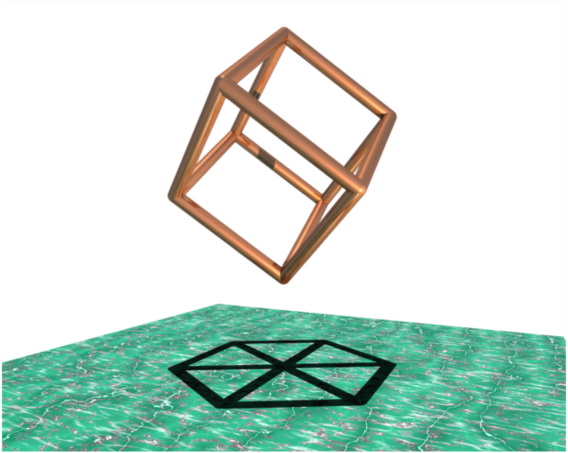
Semiregular Solids

What about its symmetries characterizes a semiregular solid?

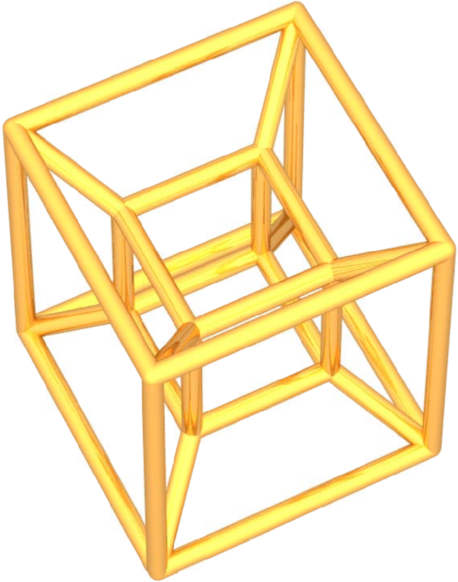
- Every face is a regular polygon, and
- The symmetry group of the semiregular solid is vertex-transitive.

Shadows of the Fourth Dimension

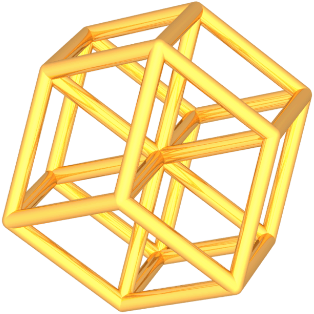
Projection of a Cube



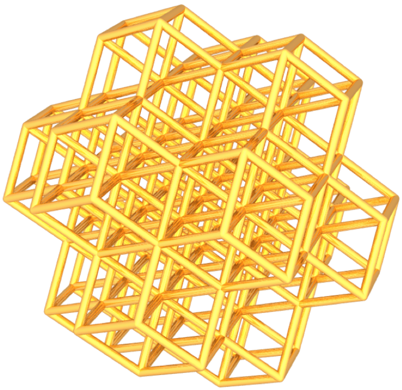
Projection of a Hypercube



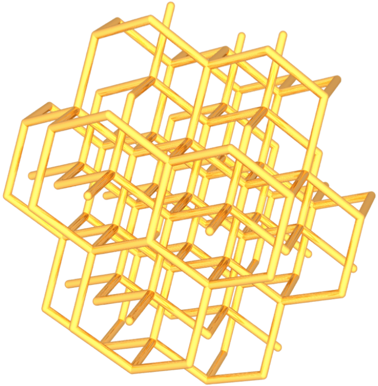
Projection of a Hypercube



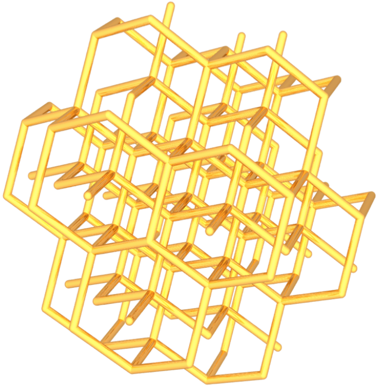
Projection of Many Hypercubes



Delete Half the Edges



Delete Half the Edges



Diamond crystal!

Ubiquity and Beauty

Ubiquity and Beauty

See accompanying powerpoint

Image Sources

Cundy and Rollett: http://www.amazon.com/Mathmatical-Models-Second-Martyn-Cundy/dp/B00KK5GFQ8/ref=sr_1_12?ie=UTF8&qid=1427234942&sr=8-12&keywords=cundy+rollett
Holden: http://www.amazon.com/Shapes-Space-Symmetry-Dover-Mathematics/dp/0486268519
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