1 CHECKER-STACKS

This game is played with several stacks of black and red checkers. You can choose any initial configuration you like. See Figure 1 for example (red checkers are drawn as white).

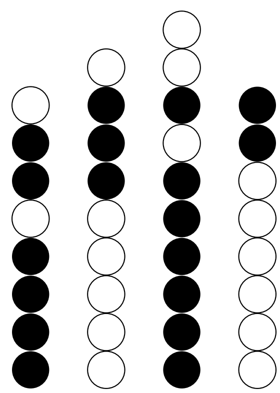

Figure 1: A Checker-Stacks Position

There are two players, Left and Right. Left goes first and moves by choosing any one stack and taking any Black checker and all checkers (if any) above it. Right moves by choosing any one stack and taking any Red checker and all checkers (if any) above it. The first player unable to move because there are no checkers of his color remaining, loses.

Problem 1.1 Experiment with the first configuration for a while. Who can win? What are some winning strategies?

Problem 1.2 Analyze the position in Figure 2.

Problem 1.3 Analyze the position in Figure 3.

Problem 1.4 Analyze the position in Figure 4.

Problem 1.5 What is the value of the position in Figure 5?

Problem 1.6 What is the value of the position in Figure 6?

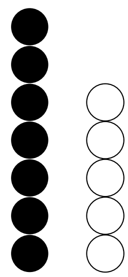

Figure 2: A Checker-Stacks Position

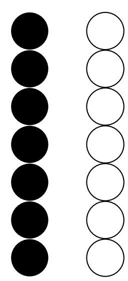

Figure 3: A Checker-Stacks Position

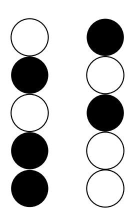

Figure 4: A Checker-Stacks Position

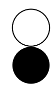

Figure 5: A Checker-Stacks Position

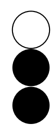

Figure 6: A Checker-Stacks Position

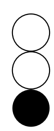

Figure 7: A Checker-Stacks Position

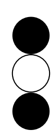

Figure 8: A Checker-Stacks Position

Problem 1.7 What is the value of the position in Figure 7?

Problem 1.8 What is the value of the position in Figure 8?

Problem 1.9 What is the value of the position in Figure 1? Who can win? How?

Problem 1.10 What position has value $\frac{1}{3}$?

Problem 1.11 Can you figure out a way of systematically determining the values of positions?

Problem 1.12 Can you figure out a way of systematically constructing positions of various values?

2 GAME OF SKI-JUMPS

Figure 9 shows a ski-slope with some skiers L and R belonging to Left and Right, respectively. In a single move, Left may move any of his skiers a

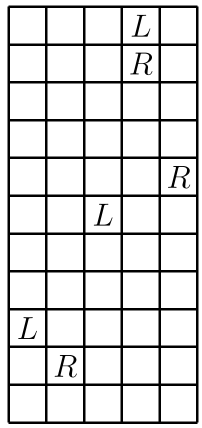

Figure 9: A Ski-Jumps Position

square or more Eastwards (to the right), or Right any of his, Westwards (to the left), provided there is no other active skier in the way. Such a move may take the skier off the slope; in this case he takes no further part in the game. No two skiers may occupy the same square of the slope. Alternatively a skier on the square immediately *above* one containing a skier of the opposing team, may jump over him onto the square immediately below, provided this is empty. A man jumped over is so humiliated that he will never jump over anyone else—in fact he is demoted from being a *jumper* to an ordinary skier, or *slipper,* and is thereafter denoted by a lower case letter r or ℓ.

No other move is permitted in this game, so that when all the skiers belonging to one of the players have left the ski-slope, that player cannot move, and a player who cannot move when it is his turn to do so, loses the game.

Problem 2.1 Experiment with the first position for a while. Who can win? How?

Problem 2.2 Analyze the position in Figure 10.

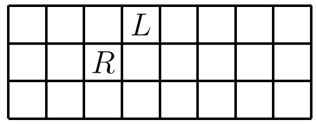

Figure 10: A Ski-Jumps Position

Problem 2.3 Analyze the position in Figure 11.

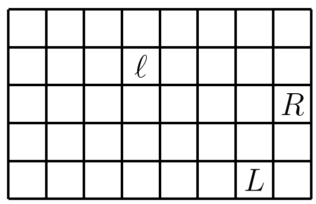

Figure 11: A Ski-Jumps Position

Problem 2.4 Analyze the position in Figure 12.

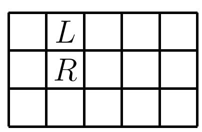

Figure 12: A Ski-Jumps Position

Problem 2.5 What is the value of the position in Figure 13?

Problem 2.6 What is the value of the position in Figure 14?

Problem 2.7 Determine the values of all positions that can arise from Figure 15.

		L		
			R	

Figure 13: A Ski-Jumps Position

L				
	R			

Figure 14: A Ski-Jumps Position

L				
				R

Figure 15: A Ski-Jumps Position

Problem 2.8 From the results of the previous problem, verify that a potential jump is worth half a move only if it is either imminent or the two players are the same distance from the central column. It is worth a whole move (just as if it were a sure thing) if the potential jumper is nearer to the central column than the jumpee and worth nothing (just as if it were impossible) otherwise.

Problem 2.9 What is the value of the position in Figure 9. Who will win and how?

3 GAME OF TOADS-AND-FROGS

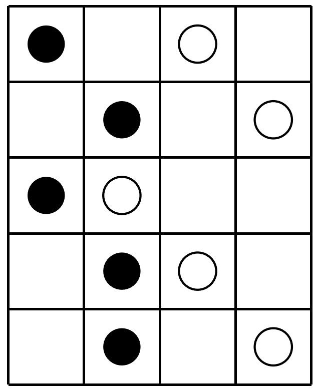

Figure 16: A Toads-and-Frogs Position

Left has trained a number of Toads and Right a number of Frogs to play the following game. Each player may persuade one of his creatures either to move one square or to jump over an opposing creature, onto an empty square. Toads move only Eastward, Frogs only to the West (*to*ads *to, fro*gs *fro*). The game is to be played according to the normal play rule that a player unable to move loses. Toads are represented by black circles, Frogs by white.

Problem 3.1 Experiment with the first position for a while. How can win? What are the best moves?

Problem 3.2 Determine the values of all positions arising from Figure 17. Then try the previous problem again.

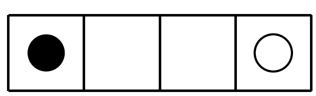

Figure 17: A Toads-and-Frogs Position

4 WHAT IS A GAME?

Our games of Checker-Stacks and Ski-Jumps are typical of almost all discussed in the first part of this course in that:

1. There are just two players, often called Left and Right.
2. There are several, usually finitely many, *positions,* and often a particular *starting position.*
3. There are clearly defined *rules* that specify the *moves* that either player can make from a given position to its *options.*
4. Left and Right move alternately, in the game as a whole.
5. In the *normal play* convention a player unable to move loses.
6. The rules are such that Play will always come to an end because some player will be unable to move. This is called the *ending condition.* So there can be no games which are drawn by repetition of moves.
7. Both players know what is going on, i.e., there is *complete information.*
8. There are no *chance moves* such as rolling dice or shuffling cards.

The reader should see how far his own favorite games satisfy these conditions.

Problem 4.1 For the following games, which of the above conditions are satisfied and which are not? Why?

1. Tic-Tac-Toe
2. $3 \times 3 \times 3$ Tic-Tac-Toe
3. Go-Moku
4. Chess
5. Snakes-and-Ladders
6. Backgammon

7. Battleships
8. Kriegspiel
9. Three-Finger Morra
10. Scissors-Paper-Stone
11. Monopoly
12. Solitaire
13. Poker
14. Bridge
15. Basketball
16. Nim
17. Dots-and-Boxes
18. Go

5 CUTCAKE

Mother has just made the oatmeal cookies shown in Figure 18. She hasn't yet

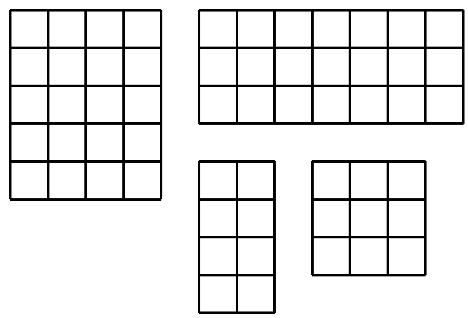

Figure 18: A Cutcake Position

broken them up into little squares, although she has scored them along the lines indicated. Rita and her brother Lefty decide to play a game breaking them up. Lefty will cut any rectangle into two smaller ones along one of the North-South lines, and Rita will cut some rectangle along an East-West line. when one of the children is unable to move, the game ends, and that child is the loser.

Problem 5.1 Determine the values of single rectangles of dimensions 1×1, 1×2, 1×3, etc., and of dimensions 2×1, 3×1, etc.

Problem 5.2 Determine the values of single rectangles of dimensions 2×3 and 2×4.

Problem 5.3 Determine the values of single rectangles of dimension $m \times n$ for various values of m and n and try to detect the general pattern. Then analyze the position of Figure 18.

6 MAUHIN'S CUTCAKE

This is like Cutcake, except Lefty's move is to divide one cake into *any* number of *equal* pieces, using only vertical cuts, while Rita does likewise, but with horizontal cuts. Once again the cuts must follow Mother's scorings, so that all dimensions will be whole numbers.

Problem 6.1 Determine the values of single rectangles of dimension $m \times n$ for various values of m and n and try to detect the general pattern.

7 HICKERSON'S CUTCAKE

This is like Cutcake, except Lefty must make exactly v vertical cuts at each turn, and Rita h horizontal ones.

Problem 7.1 Choose some numbers v and h, and determine the values of single rectangles of dimension $m \times n$ for various values of m and n and try to detect the general pattern.

8 BLUE-RED HACKENBUSH

This game is played with a picture drawn with blue and red edges which is based on a horizontal line (the *ground*). We shall call the two players Left and Right. Left moves by deleting any blue edge, together with any edges that are no longer connected to the ground, and Right moves by deleting a Red edge in a similar way. (Play it on a blackboard if you can, because it's easier to rub the edges out.) The first player unable to move is the loser.

Problem 8.1 Determine the values of some simple starting positions.

9 HACKENBUSH HOTCHPOTCH

This game is played as Blue-Red Hackenbush except that there may also be some *green* edges, which either player may chop. But blue edges are still reserved for Left, and red ones for Right and we continue to use the normal play rule, that when you can't move, you lose.

Problem 9.1 Analyze the starting position consisting of a flower with green stem, three blue petals, and two red petals. Show that it is a fuzzy position.

Problem 9.2 Analyze the starting position consisting of two of the above flowers.

10 SUMS OF ARBITRARY GAMES

Problem 10.1 Prove that if G and H are greater than or equal to 0, so is $G + H$.

Problem 10.2 Prove that if G is positive or fuzzy and H is positive or zero, then $G + H$ is positive or fuzzy.

Problem 10.3 Analyze the house and garden position of Hackenbush Hotchpotch.

Problem 10.4 Analyze the greenwood trees position of Hackenbush Hotchpotch.

Problem 10.5 Prove that for any game G, the game $G + (-G)$ is a zero game.

Problem 10.6 Prove that the value of the flower of Problem 9.1 is greater than all negative numbers and less than all positive numbers.

Problem 10.7 Prove that any Hackenbush Hotchpotch position in which all the ground edges are green has a value which lies strictly between all negative and all positive numbers.

Problem 10.8 Analyze the value of the small house of Hackenbush Hotchpotch.

11 COL

This game is played with a real or fictitious map of several countries. Each player, when it is his turn to move, paints one region of the map, Left using the color blue and Right using red. No two regions having a common frontier edge may be painted the same color. Whoever is unable to paint a region loses.

Problem 11.1 Analyze the map of Figure 19.

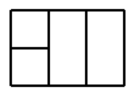

Figure 19: A Col Position

Problem 11.2 Analyze the value of the Hackenbush position consisting of only one green edge connected to the ground.

Problem 11.3 Analyze the value of the game which is the sum of the two games of Problems 11.1 and 11.2.

Problem 11.4 Experiment with other Col positions.

12 NIM

This game is played with a number of heaps of checkers. The move is to remove any positive number of checkers from any one heap. The winner is the person to take the very last counter of all.

Problem 12.1 Experiment with the following initial positions: (1,1), (2,2), (3,3), (n, n), (1,2,3), (1,4,5), (2,4,6), (3,5,7).

Problem 12.2 Show that a single heap of size n has value $*n$.

Problem 12.3 Evaluate the values of the positions given in Problem 12.1.

Problem 12.4 Prepare a nim-addition table.

13 CHILDISH HACKENBUSH

We call a Hackenbush picture *childish* because every edge is connected to the ground, perhaps via other edges. The rule in ordinary Hackenbush is that edges which might make a picture non-childish are deleted as soon as they arise. However, in Childish Blue-Red Hackenbush you are only allowed to take edges which leave all the others connected to the ground; nothing may fall off.

Problem 13.1 Try to determine the Childish Hackenbush values and winning strategies of some initial pictures.

14 SEATING COUPLES

Consider a round dining table around which Left and Right are taking turns to seat couples for a dinner. Left prefers to seat a lady to the *left* of her partner, while Right thinks it proper only to seat her to the *right.* No gentleman may be seated next to a lady other than his own partner. The player, Left or Right, who first finds himself unable to seat a couple, has the embarrassing task of turning away the remaining guests, and so may be said to *lose.*

Of course the rules have the effect of preventing either player Left or Right from seating two couples in four adjacent chairs, for then the gentleman from one of his two couples will be next to the lady from the other. So when either player seats a couple, he effectively reserves the two seats on either side for the use of his opponent only. So after the game has started, the available chairs will form rows of three types:

1. LnL, a row of n empty chairs between two of Left's guests,

2. RnR, a row of n empty chairs between two of Right's guest, and

3. LnR or RnL, a row of n empty chairs between one of Left's guests and one of Right's.

Problem 14.1 Determine the values of the three above types of rows for various values of n and try to detect a general pattern. Then explain how to win the game for various sizes of tables.

15 MISCELLANEOUS

Problem 15.1 Show that it is possible that two games D and G which individually satisfy the ending condition might have a sum $D + G$ that does not. This shows that we must add the condition that in no game is there any infinite sequence $G \to G_1 \to G_2 \to \cdots$ of legal moves, *alternating or not.*

16 POKER-NIM

This game is played with heaps of Poker-chips. Just as in ordinary Nim, either player may reduce the size of any heap by removing some of the chips. But now we allow a player the alternative move of increasing the size of some heap by adding to it some of the chips he acquired in earlier moves. These two kinds of moves are the only ones allowed.

Problem 16.1 Play this game with various starting positions and try to find some winning strategies.

17 NORTHCOTT'S GAME

Northcott's game is played on a checkerboard which has one black and one white piece on each row, as in Figure 20.

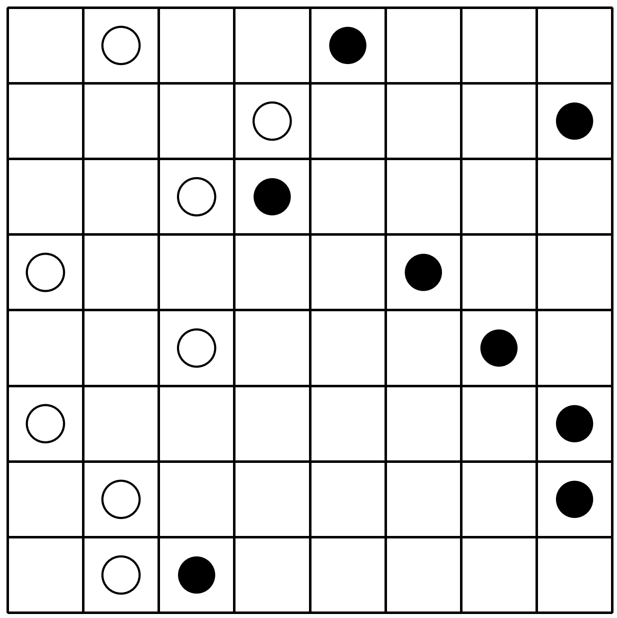

Figure 20: A Position in Northcott's Game

You may move any piece of your own color to any other empty square in the same row, provided you do not jump over your opponent's piece in that row. If you can't move (because all your pieces are trapped at the side of the board by your opponent's), you lose.

Problem 17.1 Play this game with various starting positions and try to find some winning strategies. Then evaluate the position in Figure 20.

18 BLACK KNIGHT

The Black Knight has, from any position on the chessboard, the moves shown in Figure 21. The game ends when the Knight is on one of the four home

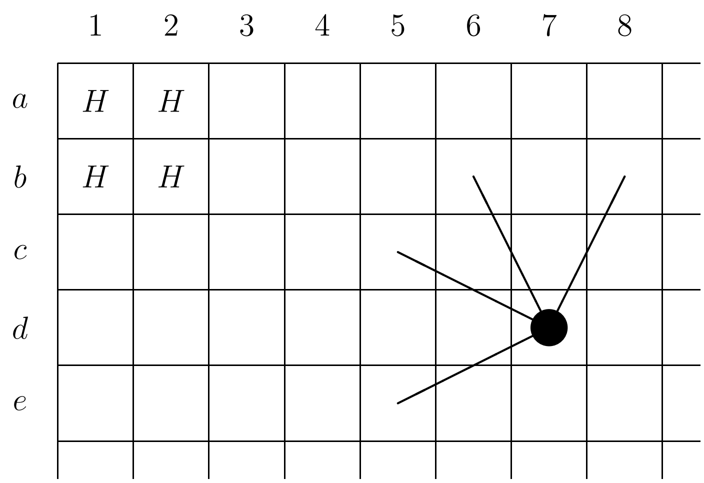

Figure 21: Legal Moves for the Black Knight

squares.

Problem 18.1 Determine the values of various starting positions for the Black Knight. Try to detect a general pattern.

Problem 18.2 Discuss the game of Black Knights which is the same as Black Knight except that any number of knights may be placed on the board, any number of knights may occupy the same square at the same time, and the goal is to bring them all into the group of four home squares.

19 WHITE ROOK AND WHITE ROOKS

This is the same as Black Knight and Black Knights, except that each knight is replaced by a White Rook which can only move North or West, even jumping over other rooks. The goal is to bring the rook(s) into the upper left-hand corner.

Problem 19.1 Determine the values of various starting positions for the White Rook. Try to detect a general pattern.

20 WHITE QUEEN AND WHITE QUEENS

This is the same as Black Knight and Black Knights, except that each knight is replaced by a White Queen which can only move North, West, or Northwest, even jumping over other queens. The goal is to bring the queen(s) into the upper left-hand corner.

Problem 20.1 Determine the values of various starting positions for the White Queen. Which ones have value 0? Try to detect a general pattern.