Visualizing Mathematics: Some Notes

Carl W. Lee
University of Kentucky

Spring 2014

Contents

1 Visualizing

In this seminar we will be looking at various interactions between mathematics and visualization. Some examples are:

1. Illustrating mathematical structures
2. Illustrating mathematical theorems
3. Illustrating mathematical proofs
4. Simulations
5. Using mathematics to create images
6. Using mathematics to create animations

Some examples of visualization media:

1. Physical models
2. Computer models
3. Physical movement
4. Music and visual art
5. Literature
6. Videos

This set of notes will evolve dynamically as the course progresses. Here is the course website: `www.ms.uky.edu/~lee/visual05/visual05.html`.

2 Examples of Visualizations and "Snapshots"

Look at some of the examples on the course website.

3 Some Polyhedron Problems

Polyhedra provide fruitful fodder for physical and virtual mathematical models. Here are some problems to get started. It will be helpful to have construction materials like Polydron.

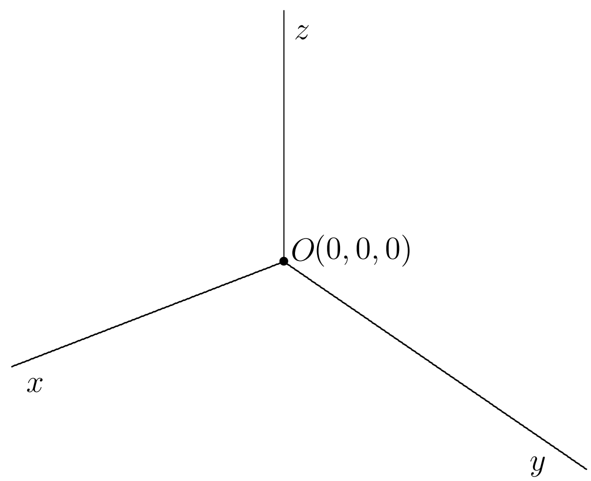
1. Try to find (construct) convex polyhedra such that every face is an equilateral triangle and adjacent triangles do not lie in the same plane. (The polyhedron is convex if the entire line segment joining any two points of the polyhedron lies within the polyhedron.) These are the *deltahedra*. What do you notice about the numbers of faces of the polyhedra constructed? Make some conjectures. Try to find some proofs.

2. Consider clusters of regular polygons that fit together around a common vertex (corner), but with a total angle sum of less than 360 degrees. Let's call these *space clusters*. For example, the cluster of three squares, which can denote $(4, 4, 4)$, makes a total angle of only 270 degrees. Of course, this cluster can be extended so that the same cluster appears at each vertex, eventually closing up to make a cube. Some space clusters cannot be extended to create a polyhedron. The space cluster $(4, 4, 4,)$ consists of only one type of polygon, as opposed to, say $(3, 4, 3, 4)$—"triangle, square, triangle, square"—which consists of more than one type of polygon. Find as many space clusters as you can that can be extended to enclose a polyhedron. If the cluster contains only one type of polygon, the resulting polyhedron is called a *Platonic solid* or *regular polyhedron*. If the cluster contains a mixture of two or more polygons, the resulting polyhedron is called an *semiregular polyhedron*. There are two infinite families of semiregular polyhedra—the *prisms* and the *antiprisms*. Besides theses there are just 13 others, known as the *Archimedean solids*.

3. Try to find coordinates for the vertices of some of these polyhedra.

4 Cartesian Coordinates

You are probably already familiar with two- and three-dimensional Cartesian coordinate systems. The typical view of the three-dimensional coordinate system is usually something like this:



Points, or locations, are ordered triples (x, y, z) of numbers. To find the point associated with the triple, begin at the origin $(0, 0, 0)$, move x units in the direction of the positive x-axis (move backwards if x is negative), y units in the direction of the positive y-axis, and z units in the direction of the positive z-axis.

It is sometimes useful to specify the location of a point with respect to two others. For example, if we are given the coordinates of points A and B and t is a real number, then $C = (1-t)A + tB$ is a point on the line through A and B. This is the *parametric* form of the line. If $t = 0$ then we get the point A. If $t = 1$ then we get the point B. If $t = 0.5$ then we get the midpoint of the segment joining A and B. If $t = 1/3$ then we get the point one third of the way from A to B. If $t = 3$ then we get the point three times as far from A than B.

The distance between two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ is given by the *distance formula*

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

This is really a manifestation of the Pythagorean Theorem, which may be more obvious to you if you look at the distance formula in the plane.

Another useful formula, used for solving quadratic equations $ax^2 + bx + c = 0$, is the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

5 Introduction to POV-Ray

POV-Ray is powerful, yet free, ray-tracing software that can produce beautiful three-dimensional images. Download the software from the website `www.povray.org`. It is available for Windows as well as the Mac operating systems. The syntax can be intimidating, but with practice one can fairly quickly begin producing some nice images. Download some of the examples from the course website, and study them to see what the instructions are doing.

After POV-Ray is installed and started up, do the following to render a test scene:

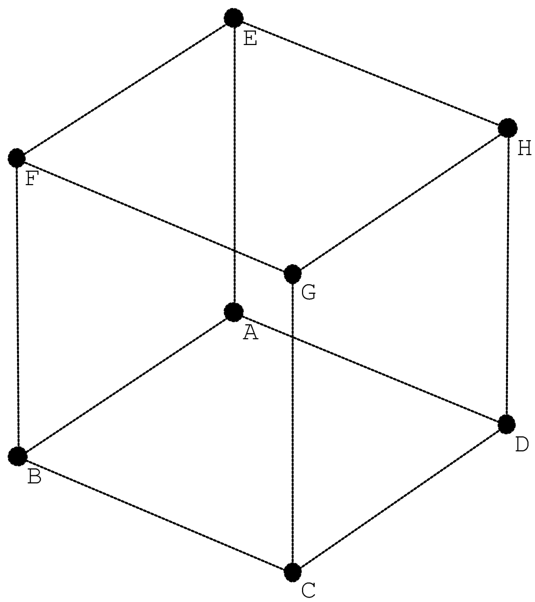
1. Go to the course website, select "POV-Ray Examples", and then select "Source file" under "Sphere". Use "Edit—Select All" and then "Edit—Copy".
2. Within POV-Ray select "File—New File". Then select "Edit—Paste".
3. Select "File—Save" to save the file somewhere on your computer.
4. Select your desired picture size, such as "[512x384, AA 0.3]" from the pulldown menu.
5. Select "Render—Start Rendering (Go!)" to create the picture. If all goes well, you should see a resulting image.

We'll come to animations later.

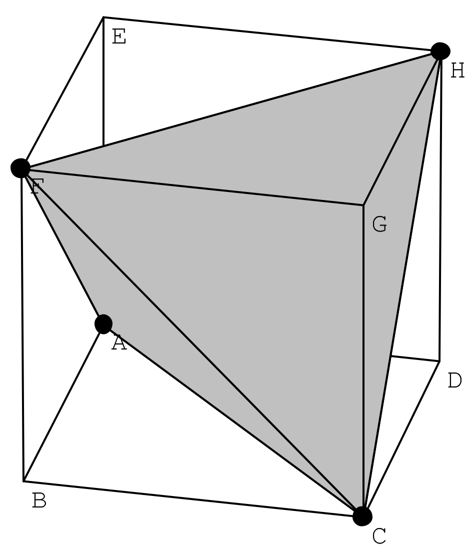
6 Coordinates for the Platonic Solids

Let's form a cube so that the two opposite vertices, A and G, below, have coordinates $A = (-1, -1, -1)$ and $G = (1, 1, 1)$. Determine the coordinates of the remaining vertices.

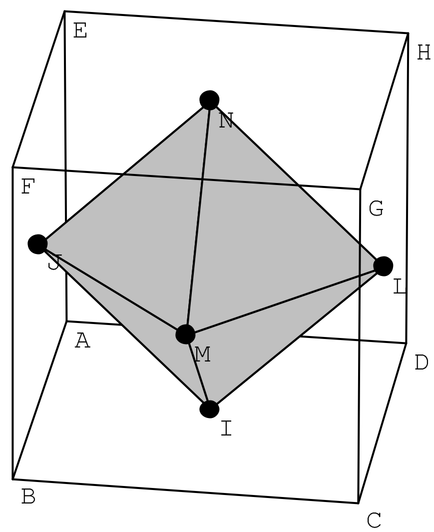
You can view this picture as a Wingeom file cube.wg3. If you have a computer with a Windows operating system go to the website http://math.exeter.edu/rparris/wingeom.html and download Wingeom (click "Wingeom" in the upper left corner). Click the downloaded file to install it. Then download the file cube.wg3 from the course website. Start the program Wingeom. Select "Window→3-dim". Select "File→Open". Select the file cube.wg3. You can then use the arrow keys to turn it around.



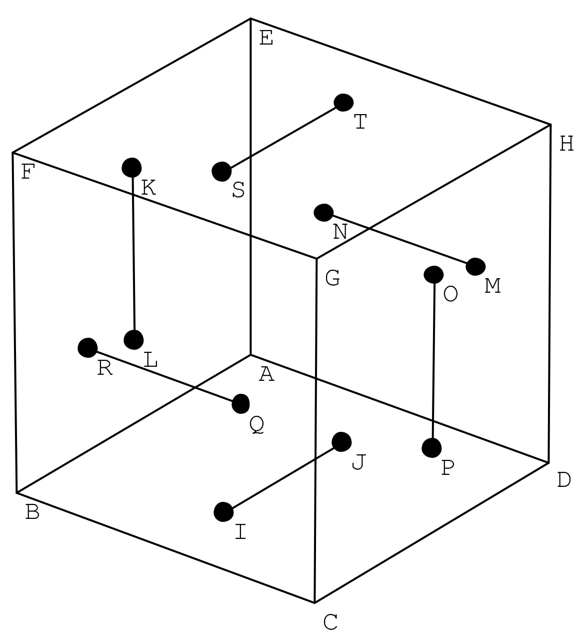
Now you can construct a regular tetrahedron by using "every other" vertex of the cube. Download and view the file tetrahedron.wg3.



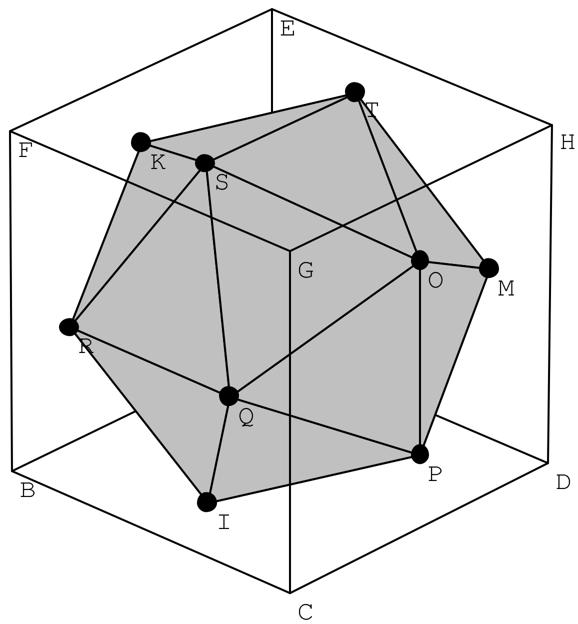
The vertices of an octahedron can be taken to be the centers of the six squares of the cube. The following comes from the file octahedron.wg3.



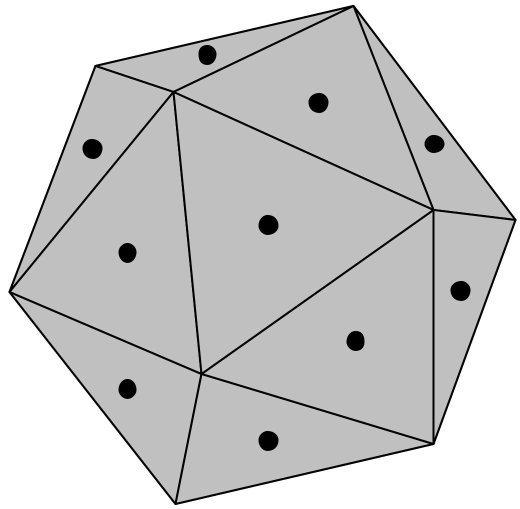
The icosahedron is trickier. First, center six line segments in each of the six faces of the cube as shown below. For example, the coordinates of the points I and J will be $I = (t, 0, -1)$ and $J = (-t, 0, -1)$, where t is a positive number between 0 and 1 yet to be determined. Similarly, the coordinates of the points K and L will be $K = (0, -1, t)$ and $L = (0, -1, -t)$ for the same value of t. The following comes from the file preicosahedron.wg3.



The goal is to determine the value of t so that when the points are joined as shown below, each of the resulting triangles is equilateral. Use algebra to calculate the exact value of t. The picture comes from the file icosahedron.wg3. When you open this file from Wingeom, you can move the "# slider" to observe the effect of changing t.



Once you have the coordinates of the vertices of the icosahedron, the vertices of the dodecahedron can be taken to be the centers of each of the icosahedron's triangles. You can get the coordinates of such a center point by averaging each of the coordinates of the vertices of the triangle. The following comes from the file icosahedron2.wg3.



Index