Worksheet # 2

Recursive Sequences

Find the next three terms of the following sequences given both the first term and the recursive formula. Assume \( n = 1 \).

1. \( a_1 = 5 \) and \( a_{n+1} = 3a_n \).
2. \( a_1 = -3 \) and \( a_{n+1} = 2a_n \).
3. \( a_1 = 2 \) and \( a_{n+1} = 3a_n - 4 \).
4. \( a_1 = 16 \) and \( a_{n+1} = \frac{a_n}{4} \).

By writing down the first 4 terms using the \( n \)th term, find the recursive formula that defines each sequence.

5. \( a_n = 2n - 1 \).
6. \( a_n = 3n - 2 \).

Find the next 4 terms of these recursively defined sequences. Assume \( n \geq 2 \).

7. \( a_1 = 6, a_2 = 2, \) and \( a_{n+1} = a_n a_{n-1} \).
8. \( a_1 = 1, a_2 = -3, \) and \( a_{n+1} = 3a_n + 2a_{n-1} \).
9. \( a_1 = 3 \) and \( a_{n+1} = 5a_n - 11 \).

10. Consider the recursive sequence defined by \( a_1 = 2, \) \( a_{n+1} = 1 - \frac{1}{a_n} \). Find the first 5 terms of the sequence. Find the limit of the sequence, if it exists.

11. Consider the recursive sequence:

\[
a_1 = 3 \text{ and } a_{n+1} = \frac{3}{7}(a_n + 6).
\]

(a) Compute decimal approximations for \( a_2 \) to \( a_6 \).

(b) Your data should indicate that the sequence is monotonic increasing. Prove this by induction.

(c) By part (b), \( a_1 = 3 \) is automatically a lower bound for the sequence. Prove by induction that 5 is an upper bound for the sequence.

(d) Find the limit of this sequence.
12. For the given recursive sequence, find the value of the limit, if it exists. \( a_1 = 2, \ a_{n+1} = 43a_n \).

13. Write down the first 3 terms of the sequence defined by \( a_1 = 10 \) and \( a_{n+1} = 12 - a_n \). Does this sequence have a limit? If so, what is it.

14. Consider the recursive sequence

\[
a_1 = 11 \text{ and } a_{n+1} = \frac{4}{9}(a_n + 7)
\]

(a) Compute decimal approximations for \( a_2 \) to \( a_6 \).
(b) Your data should indicate that the sequence is monotonic decreasing. Prove this by induction.
(c) By part (b), \( a_1 = 11 \) is automatically an upper bound for the sequence. Prove by induction that 0 is a lower bound for the sequence.
(d) Find the limit of this sequence.

15. Consider the recursive sequence:

\[
a_1 = 7 \text{ and } a_{n+1} = 5 - \frac{3}{a_n}
\]

(a) Compute decimal approximations for \( a_2 \) to \( a_6 \).
(b) Your data should indicate that the sequence is monotonic decreasing. Prove this by induction.
(c) By part (b), \( a_1 = 7 \) is automatically an upper bound for the sequence. Prove by induction that 4 is a lower bound for the sequence. Why would it be a very bad idea to prove that 0 is a lower bound for the sequence? Be sure you know the properties of inequalities!
(d) Find the limit of this sequence.

16. Consider the recursive sequence:

\[
a_1 = 2 \text{ and } a_{n+1} = \frac{1}{3-a_n}
\]

(a) Compute \( a_2 \) to \( a_6 \).
(b) Your data should indicate that the sequence is monotonic decreasing. Prove this by induction.
(c) By part (b), \( a_1 = 2 \) is automatically an upper bound for the sequence. Prove by induction that 0 is a lower bound for the sequence.
(d) Find the limit of this sequence.
17. Consider the recursive sequence:

\[ a_1 = 1 \text{ and } a_{n+1} = 1 + \frac{1}{1 + a_n}. \]

The limit

\[ \lim_{n \to \infty} a_n = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \]

is called a continued fraction.

(a) Compute \(a_2\) to \(a_6\).

(b) Prove by induction that \(2\) is an upper bound for the sequence.

(c) Assume the limit of this sequence exists. Find the limit of this sequence.

18. Consider the recursive sequence:

\[ a_1 = 1 \text{ and } a_{n+1} = \sqrt[3]{a_n + 6} \]

(a) Compute \(a_2\) to \(a_6\).

(b) Your data should indicate that the sequence is monotonic increasing. Prove this by induction.

(c) Prove that \(0 \leq a_n \leq 2\) for all \(n \geq 1\).

(d) Find the limit of this sequence.

19. Let \(a_1 = x > 0\) and let \(a_{n+1} = \sqrt{x + a_n}\). Then we can interpret.

\[ \lim_{n \to \infty} a_n = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}. \]

Find \(\lim_{n \to \infty} a_n\). When does this limit exist?