MA109, Activity 10: Systems of Equations (Section 6.1, pp. 460-466) **Date:** ____________

Systems of Linear Equations in Two Variables (Section 6.2, pp. 469-474)

`Today's Goal:`	`We learn how to solve systems of two equations in two unknowns. They often occur when we use equations to solve problems in the sciences and in many other areas. We learn two different methods of solving such systems: By substitution and by elimination. In particular, we focus on systems of two linear equations in two variables.`
`Assignments:`	**Homework (Sec. 6.1):** `# 1, 5, 9, 13, 17, 19, 27, 35, 47, 51 (pp. 466-468).` **Homework (Sec. 6.2):** `# 1, 5, 9, 13, 27, 33, 43, 47, 51, 57 (pp. 474-476).`

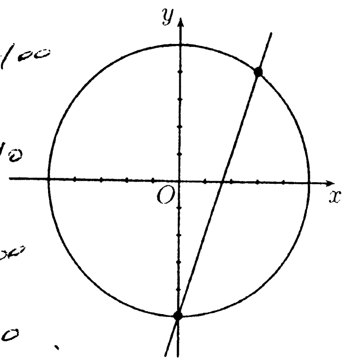
► **Systems of Equations and Their Solutions:**

A **system of equations** is a set of equations that involve the same variables. A **solution** of a system is an assignment of values for the variables that makes each equation in the system true. To **solve** a system means to find all solutions of the system .

Example 1: Show that $(0, -10)$ and $(6, 8)$ are solutions of the system

$$\begin{cases} x^2 + y^2 = 100 \\ 3x - y = 10 \end{cases}$$

$(0,-10)$ is a solution since $\begin{cases} 0^2+(-10)^2=100 \\ 3\cdot 0-(-10)=10 \end{cases}$



$(6,8)$ is a solution since $\begin{cases} 6^2+8^2=100 \\ 3\cdot 6-8=10 \end{cases}$.

► **Elimination Method:** In the elimination method we try to combine the equations using sums or differences so as to eliminate one of the variables.

1. **Adjust the Coefficients:** Multiply one or more of the equations by appropriate numbers so that the coefficient of one variable in one equation is the negative (or opposite) of its coefficient in the other equation.
2. **Add the Equations:** Add the two equations to eliminate one variable, then solve for that variable.
3. **Back-Substitute:** Substitute the value(s) you found in Step 2 back into one of the original equations, and solve for the remaining variable.

Example 2: Find all solutions of the system $\begin{cases} 4x - 3y = 11 \cdots ① \\ 8x + 4y = 12 \cdots ② \end{cases}$

How many solutions does the system have?

① ×2 : $2(4x-3y)=2\cdot 11$

$\Rightarrow 8x-6y=22 \cdots ①'$

①'−② : $-10y=10$

$\Rightarrow y=-1$

Substitute y in ① by -1 :

$4x+3=11$

$x=2$.

$\Rightarrow$ The system has one solution.

$\begin{cases} x=2 \\ y=-1 \end{cases}$.

Example 3: Find all solutions of the system $\begin{cases} 2x^2 + 4y = 13 & \cdots ① \\ x^2 - y^2 = \dfrac{7}{2} & \cdots ② \end{cases}$

How many solutions does the system have?

$② \times 2: \quad 2x^2 - 2y^2 = 7 \quad \cdots ②'$

$① - ②': \quad 4y + 2y^2 = 6$

$\Rightarrow 2y^2 + 4y - 6 = 0$

$\Rightarrow y^2 + 2y - 3 = 0$

$(y+3)(y-1) = 0$

$\Rightarrow y = -3$ or 1.

Substitute y in ① by -3.

$2x^2 - 12 = 13$

$\Rightarrow x^2 = \dfrac{25}{2}$

$\Rightarrow x = \pm\dfrac{5}{\sqrt{2}}$

Substitute y in ① by 1,

$2x^2 + 4 = 13$

$\Rightarrow x^2 = \dfrac{9}{2}$

$\Rightarrow x = \pm\sqrt{\dfrac{9}{2}} = \pm\dfrac{3}{\sqrt{2}}$

So the system has four solutions

$\begin{cases} x = \dfrac{5}{\sqrt{2}} \\ y = -3 \end{cases}$, $\begin{cases} x = -\dfrac{5}{\sqrt{2}} \\ y = -3 \end{cases}$, $\begin{cases} x = \dfrac{3}{\sqrt{2}} \\ y = 1 \end{cases}$, $\begin{cases} x = -\dfrac{3}{\sqrt{2}} \\ y = 1 \end{cases}$

► **Systems of Linear Equations in Two Variables:**

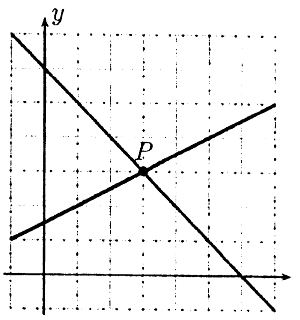
A system of two linear equations in two variables has the form $\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$

where a_1, a_2, b_1, b_2, c_1, and c_2 are numbers. For linear systems the elimination method is usually the easiest one. We will see later, though, that one could also use the substitution method.

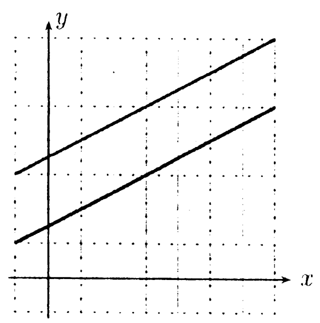
The graph of a linear system in two variables is a pair of lines. Thus, from a graphic point of view, to solve the system means that we must find the intersection point(s) of the lines.

Number of Solutions of a Linear System in Two Variables:
For a system of linear equations in two variables, exactly one of the following is true:

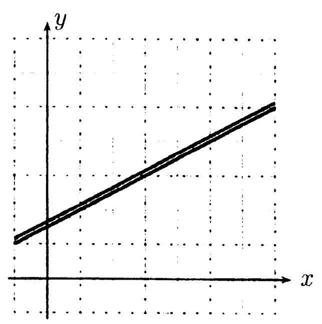
1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.



Linear system with one solution. Lines intersect at a single point.



Linear system with no solution. Lines are parallel, so they do not intersect.



Linear system with infinitely many solutions. Lines coincide.

Example 4: Solve the system $\begin{cases} 3x - 6y &= 12 \\ 4x - 8y &= 15 \end{cases}$
How many solutions does the system have?

Multiply top by $-\frac{4}{3}$

$$-4\cancel{x} + 8\cancel{y} = -16$$
$$4\cancel{x} - 8\cancel{y} = 15$$

$$0 = -1$$

FALSE!

No Solutions

Example 5: Solve the system $\begin{cases} 2x + y &= 11 \\ x - 2y &= 4 \end{cases}$
How many solutions does the system have?

Mult. bottom by -2

$$2\cancel{x} + y = 11$$
$$-2\cancel{x} + 4y = -8$$

$$5y = 3$$
$$\boxed{y = \frac{3}{5}}$$

$$x - 2y = 4$$
$$x - 2\left(\frac{3}{5}\right) = 4$$
$$x - \frac{6}{5} = 4$$
$$x = 4 + \frac{6}{5}$$
$$\boxed{x = \frac{26}{5}}$$

$$\left(\frac{26}{5}, \frac{3}{5}\right)$$

Example 6: Show that the system $\begin{cases} 12x + 15y &= -18 \\ 4x + 5y &= -6 \end{cases}$ has infinitely many solutions and express them in the ordered pair (parametric) form given in Example 6.2.3 (page 471 of the textbook).

Multiply bottom by -3

$$12x + 15y = -18$$
$$-12x - 15y = 18$$

$$0 = 0$$

Infinite Solutions

Parametric Form

$$4x + 5y = -6$$
$$5y = -4x - 6$$
$$y = \frac{-4x - 6}{5}$$

So if $x = a$, $y = \frac{-4a-6}{5}$

All solutions look like $\left(a, \frac{-4a-6}{5}\right)$

▶ **Substitution Method:** In the substitution method we start with one equation in the system and solve for one variable in terms of the other variable. The following box describes the procedure.

1. **Solve for One Variable:** Choose one equation and solve for one variable in terms of the other variable.
2. **Substitute:** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
3. **Back-Substitute:** Substitute the value(s) you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

Example 7: Find all solutions of the system $\begin{cases} 2x + y = 7 & \text{--- ①} \\ x + 2y = 2 & \text{--- ②} \end{cases}$

How many solutions does the system have?

Solve ① for y: $y = 7 - 2x$.

Substitute: $x + 2(7-2x) = 2$

$\Rightarrow -3x = -12$

$\Rightarrow x = 4$

Back-Substitute: $2 \cdot 4 + y = 7 \Rightarrow y = -1$

The system has one solution: $\begin{cases} x = 4 \\ y = -1 \end{cases}$

Example 8: Find all solutions of the system $\begin{cases} x^2 + y = 9 & \text{--- ①} \\ x - y + 3 = 0 & \text{--- ②} \end{cases}$

How many solutions does the system have?

Solve ① for y: $y = 9 - x^2$

Substitute: $x - (9 - x^2) + 3 = 0$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x+3)(x-2) = 0$

$\Rightarrow x = -3, 2$

Backsubstitute: $-3 - y + 3 = 0 \Rightarrow y = 0$ or $2 - y + 3 = 0 \Rightarrow y = 5$

The system has two solutions: $\begin{cases} x = -3 \\ y = 0 \end{cases}$, $\begin{cases} x = 2 \\ y = 5 \end{cases}$.

Example 9: Find all solutions of the system $\begin{cases} xy = 24 & \text{--- ①} \\ 2x^2 - y^2 + 4 = 0 & \text{--- ②} \end{cases}$

How many solutions does the system have?

From ①: $y = \frac{24}{x}$.

Substitute: $2x^2 - \left(\frac{24}{x}\right)^2 + 4 = 0$

$\Rightarrow 2x^4 - 24^2 + 4x^2 = 0$

$x^4 + 2x^2 - 288 = 0$

$(x^2 + 18)(x^2 - 16) = 0$

$\Rightarrow x^2 = -18, 16$.

$\Rightarrow x^2 = 16$ (-18 is negative)

$\Rightarrow x = \pm 4$.

Back-substitute:

$4y = 24$

$\Rightarrow y = 6$

or $-4y = 24$

$\Rightarrow y = -6$

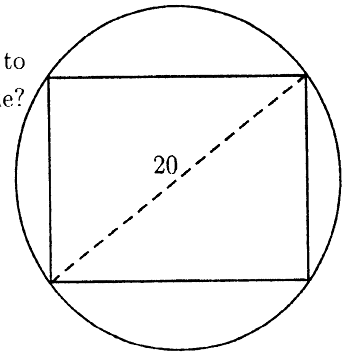
$\Rightarrow$ The system has two solutions

$\begin{cases} x = 4 \\ y = 6 \end{cases}$, $\begin{cases} x = -4 \\ y = -6 \end{cases}$.

▶ **Modeling with Systems of Equations: Guidelines**

1. **Identify the Variables.** Identify the quantities the problem asks you to find. These are usually determined by a careful reading of the question posed at the end of the problem.
2. **Express All Unknown Quantities in Terms of the Variables.** Read the problem again and express all the quantities mentioned in the problem in terms of the variables you defined in Step 1.
3. **Set Up a System of Equations.** Find the facts in the problem that give the relationships between the expressions you found in Step 2. Set up a system of equations that expresses these relationships.
4. **Solve the System and Interpret the Results.**

Example 10: A circular piece of sheet metal has a diameter of 20 in. The edges are to be cut off to form a rectangle of area 160 in^2. What are the dimensions of the rectangle?



x: the width , y: the length

$$\Rightarrow \begin{cases} x \cdot y = 160 \cdots ① \\ x^2 + y^2 = 20^2 \cdots ② \end{cases}$$

From ① : $y = \frac{160}{x}$

Substitute :

$$x^2 + \left(\frac{160}{x}\right)^2 = 20^2$$
$$x^4 + 160^2 = 400x^2$$
$$x^4 - 400x^2 + 160^2 = 0$$
$$(x^2 - 320)(x^2 - 80) = 0$$
$$x^2 = 320 \Rightarrow x = \pm\sqrt{320}$$
$$\text{or } x^2 = 80 \Rightarrow x = \pm\sqrt{80}$$

Note that $0 < x < y$.

So $x = \sqrt{80}$

Substitue x in ② by $\sqrt{80}$

$$y^2 = 320$$
$$\Rightarrow y = \pm\sqrt{320}. \text{ But } y > 0$$
$$\Rightarrow y = \sqrt{320}.$$

Example 11 (Boat Speed):

A boat on a river travels downstream between two points, 20 miles apart, in one hour. The return trip against the current takes $2\frac{1}{2}$ hours. What is the boat's speed, and how fast does the current in the river flow?

x: the boat's spead

y: the speed of the flow.

$$\Rightarrow \begin{cases} \frac{20}{x+y} = 1 \\ \frac{20}{x-y} = 2\frac{1}{2} \end{cases} \Rightarrow \begin{cases} 20 = x+y \\ 20 = 2\frac{1}{2}(x-y) \end{cases} \Rightarrow \begin{cases} x+y = 20 \\ x-y = 8 \end{cases}$$

$$\Rightarrow \begin{cases} x = 14 \\ y = 6 \end{cases}$$

Example 12 (Mixture Problem):

A chemist has two large containers of sulfuric acid solution, with different concentrations of acid in each container. Blending 300 mL of the first solution and 600 mL of the second solution gives a mixture that is 15% acid, whereas 100 mL of the first mixed with 500 mL of the second gives a $12\frac{1}{2}\%$ acid mixture. What are the concentrations of sulfuric acid in the original containers?

Let x, y be the concentrations of sulfuric acid in the two containers. Then

$$\begin{cases} 300 \cdot x + 600 y = 15\% (300 + 600) \\ 100 \cdot x + 500 y = 12.5\% (100 + 500) \end{cases}$$

$$\Rightarrow \begin{cases} 300 x + 600 y = 135 & \cdots ① \\ 100 \cdot x + 500 y = 75 & \cdots ② \end{cases}$$

$① - ② \times 3$: $-900 y = -90$

$y = 0.1$

Substitute y in ① by 0.1 : $300x + 60 = 135$

$x = 0.25$

Example 13 (Number Problem):

The sum of the digits of a two-digit number is 7.
When the digits are reversed, the number is increased by 27. Find the number.

Let x, y be the two digits. Then one number is $10x + y$, and the other is $10y + x$.

$$\Rightarrow \begin{cases} x + y = 7 \\ (10y + x) - (10x + y) = 27 \end{cases} \Rightarrow \begin{cases} x + y = 7 & \cdots ① \\ 9y - 9x = 27 & \cdots ② \end{cases}$$

$① \times 9 + ②$: $18 y = 90$

$\Rightarrow y = 5$

Backsubstitute : $x + 5 = 7 \Rightarrow x = 2$.

$$\Rightarrow \begin{cases} x = 2 \\ y = 5 \end{cases}$$

So the two numbers are 25 and 52