Today's Goal:	Perhaps the most useful mathematical idea for modeling the real world is the concept of a *function*. We explore the idea of a function and then give its mathematical definition.
Assignments:	**Homework (Sec. 3.1):** # 2,3,13,16,17,21,24,25,30,31,40,41,43,45,48,57,61 (pp. 221-224).

▶ Functions All Around Us:

In nearly every physical phenomenon we observe that one quantity depends on another. For instance

- height is a function of age;
- temperature is a function of date;
- cost of mailing a package is a function of weight;
- the area of a circle is a function of its radius;
- the number of bacteria in a culture is a function of time;
- the price of a commodity is a function of the demand for that commodity.

▶ Four Ways to Represent a Function:

We can describe a specific function in the following four ways:

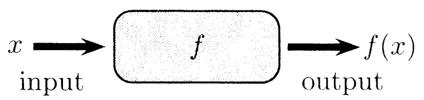
* verbally (by a description in words);
* visually (by a graph);
* algebraically (by an explicit formula);
* numerically (by a table of values).

▶ Definition of Function:

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

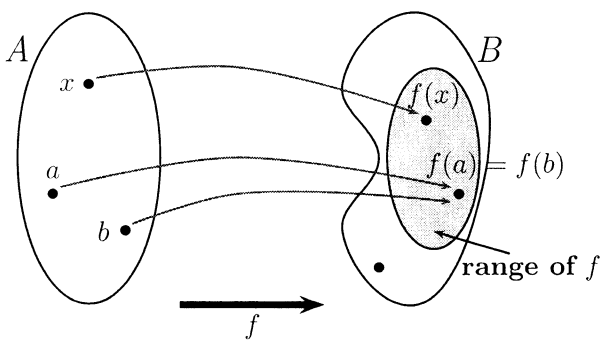
The set A is called the **domain** of f whereas the set B is called the **codomain** of f; $f(x)$ is called the **value of** f **at** x, or the **image of** x **under** f.

The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain: range of $f = \{f(x) \mid x \in A\}$.



Machine diagram of f

Arrow diagram of f



Example 1: Express the rule "square, add 9, then take the square root" in function notation.

$f(x)$

Square: x^2

add 9: x^2+9

Root: $\sqrt{x^2+9}$

$f(x) = \sqrt{x^2+9}$

▶ **Evaluating a Function:**

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. In the definition of a function the independent variable plays the role of a "placeholder". For example, the function $f(x) = 2x^2 - 3x + 1$ can be thought of as

$$f(\square) = 2 \cdot \square^2 - 3 \cdot \square + 1.$$

To evaluate f at a number (expression), we substitute the number (expression) for the placeholder.

Example 2: If $f(x) = 2x^2 - 3x + 1$ evaluate the following:

$f(4) = 2(4)^2 - 3(4) + 1$

$2(16) - 12 + 1$

$32 - 12 + 1 = 21$

$f(a) + f(b) = 2a^2 - 3a + 1 + 2b^2 - 3b + 1$

$2a^2 + 2b^2 - 3a - 3b + 2$

$f(x-1) = 2(x-1)^2 - 3(x-1) + 1$

$2(x^2 - 2x + 1) - 3x + 3 + 1$

$2x^2 - 4x + 2 - 3x + 4$

$2x^2 - 7x + 6$

$f(a+b) = 2(a+b)^2 - 3(a+b) + 1$

$2(a^2 + 2ab + b^2) - 3a - 3b + 1$

$2a^2 + 4ab + 2b^2 - 3a - 3b + 1$

Example 3: Evaluate the piecewise function $f(x) = \begin{cases} x^2 + 2x & \text{if } x \le -1 \\ x & \text{if } x > -1 \end{cases}$ at the indicated values:

$-4 \le -1$

$f(-4) = (-4)^2 + 2(-4)$

$= 16 - 8 = \boxed{8}$

$-1 \le -1$

$f(-1) = (-1)^2 + 2(-1)$

$= 1 - 2 = \boxed{-1}$

$0 > -1$

$f(0) = \boxed{0.}$

$1 > -1$

$f(1) = \boxed{1}$

Example 4: If $f(x) = 3 - 5x + 4x^2$ find:

$f(a) = 3 - 5a + 4a^2$

$f(a+h) = 3 - 5(a+h) + 4(a+h)^2$

$3 - 5a - 5h + 4(a^2 + 2ah + h^2)$

$3 - 5a - 5h + 4a^2 + 8ah + 4h^2$

$$\frac{f(a+h) - f(a)}{h} = \frac{3 - 5a - 5h + 4a^2 + 8ah + 4h^2 - (3 - 5a + 4a^2)}{h}$$

$$= \frac{\cancel{3} - \cancel{5a} - 5h + \cancel{4a^2} + 8ah + 4h^2 - \cancel{3} + \cancel{5a} - \cancel{4a^2}}{h}$$

$$= \frac{-5h + 8ah + 4h^2}{h} = -5 + 8a + 4h$$

▶ **The Domain of a Function:** The domain of a function is the set of all inputs for the function. The domain may be stated explicitly. For example, if we write

$$f(x) = 1 - x^2 \qquad -2 \le x \le 5$$

then the domain is the set of all real numbers x for which $-2 \le x \le 5$. If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain is the set of all real numbers for which the expression is defined.*

Example 5: Find the domain of each of the following functions:

- $f(x) = x^2 + 1$

All Real Numbers

- $g(t) = \dfrac{t^4}{t^2 + t - 6}$

Since we can't divide by zero

$t^2+t-6=0$

$(t+3)(t-2)=0$

$t=-3$ & $t=2$

can't be in domain

$(-\infty,-3)\cup(-3,2)\cup(2,\infty)$

- $h(u) = \sqrt{7 - 3u}$

Can't square root a negative number

$7-3u \ge 0$

$-3u \ge -7$

$u \le \frac{7}{3}$

Domain: $(-\infty, \frac{7}{3}]$

- $k(x) = \sqrt{\dfrac{x-2}{4-x}}$

$\dfrac{x-2}{4-x} \ge 0$

− + −

2 4

Crit #'s: 2, 4

T. Vals: 0, 3, 5

$\dfrac{0-2}{4-0} = \dfrac{-2}{4} = -\dfrac{1}{2}$

$\dfrac{3-2}{4-3} = \dfrac{1}{1} = 1$

$\dfrac{5-2}{4-5} = \dfrac{3}{-1} = -3$

$\boxed{x \ne 4}$

Solution and Domain: $[2,4)$

Example 6 (Torricelli's Law):

A tank holds 50 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 20 minutes. The tank drains faster when it is nearly full because the pressure on the leak is greater. **Torricelli's Law** gives the volume of the water remaining in the tank after t minutes as

$$V(t) = 50\left(1 - \frac{t}{20}\right)^2 \qquad 0 \le t \le 20.$$

(a) Find $V(0)$ and $V(20)$.

(b) What do your answers to part **(a)** represent?

(c) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.

a) $V(0) = 50\left(1 - \frac{0}{20}\right)^2 = 50(1)^2 = 50$

$V(20) = 50\left(1 - \frac{20}{20}\right)^2 = 50(1-1)^2 = 50(0) = 0$

b) $V(0)$ is amount of water at beginning, $V(20)$ is when tank is empty.

c)

t	$V(t)$
0	50
5	28.125
10	12.5
15	3.125
20	0

$V(5) = 50\left(1 - \frac{5}{20}\right)^2$
$= 50\left(\frac{3}{4}\right)^2$
$= 50\left(\frac{9}{16}\right)$
$= 28.125$

$V(10) = 50\left(1 - \frac{10}{20}\right)^2$
$= 50\left(\frac{1}{2}\right)^2$
$= 50\left(\frac{1}{4}\right)$
$= 12.5$

$V(15) = 50\left(1 - \frac{15}{20}\right)^2$
$= 50\left(1 - \frac{3}{4}\right)^2$
$= 50\left(\frac{1}{4}\right)^2$
$= 50\left(\frac{1}{16}\right)$
$= 3.125$