MA109, Activity 15: Quadratic Functions; (Section 3.5, pp. 259-266) Maxima and Minima

Date: ______

Today's Goal: In this lecture we learn how to find the maximum and minimum values of quadratic functions. For a function that represents the profit in a business, we are interested in the maximum value; for a function that represents the amount of material to be used in a manifacturing process, we are interested in the minimum value.

Assignments: **Homework (Sec. 3.5):** #1,3,6,15,22,25,34,39,41,47,59,61 (pp. 266-269).

▶ Graphing Quadratic Functions Using the Standard Form:

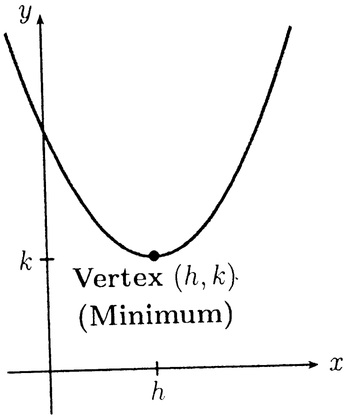
A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c,$$

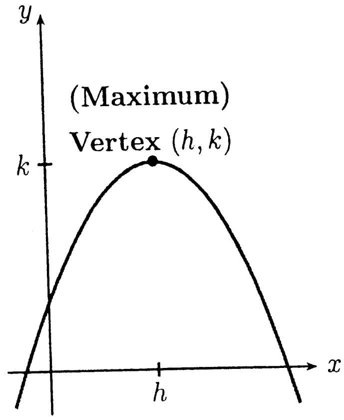
where a, b, and c are real numbers and $a \neq 0$.
The graph of any quadratic function is a parabola; it can be obtained from the graph of $f(x) = x^2$ by the methods described in Activity 14.
Indeed, by completing the square a quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x-h)^2 + k.$$

The graph of f is a parabola with vertex (h, k); the parabola opens upward if $a > 0$, or downward if $a < 0$.



$f(x) = a(x-h)^2 + k$, $a > 0$



$f(x) = a(x-h)^2 + k$, $a < 0$

▶ Maximum and Minimum Values of Quadratic Functions:

As the picture above shows:

if $a > 0$, then the **minimum value** of f occurs at $x = h$ and this value is $f(h) = k$;

if $a < 0$, then the **maximum value** of f occurs at $x = h$ and this value is $f(h) = k$.

Expressing a quadratic function in standard form helps us sketch its graph and find its maximum or minimum value. There is a **formula** for (h, k) that can be derived from the general quadratic function as follows:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

Thus:

$$\boxed{h = -\frac{b}{2a} \qquad k = \frac{4ac - b^2}{4a}}$$

If $a > 0$, then the **minimum value** is $f(-b/2a)$.

If $a < 0$, then the **maximum value** is $f(-b/2a)$.

Example 1:

Express the parabola $y = x^2 - 4x + 3$ in standard form and sketch its graph. In particular, state the coordinates of its vertex and its intercepts.

$$y = x^2 - 4x + 3 = x^2 - 4x + 4 - 4 + 3 = (x-2)^2 - 4 + 3 = (x-2)^2 - 1$$

complete the square

Thus, the vertex is $(2, -1)$

and its intercepts are $x = \frac{4 \pm \sqrt{16 - 4 \cdot 3 \cdot 1}}{2 \cdot 1} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} = 1, 3$

hence $(1,0), (3,0)$.

Example 2:

Express the parabola $y = -2x^2 - x + 3$ in standard form and sketch its graph. In particular, state the coordinates of its vertex and its intercepts.

$$y = -2x^2 - x + 3 = -2(x^2 + \tfrac{1}{2}x - \tfrac{3}{2}) = -2\left(x^2 + \tfrac{1}{2}x + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2 - \tfrac{3}{2}\right) = -2\left((x + \tfrac{1}{4})^2 - \tfrac{25}{16}\right) = -2(x + \tfrac{1}{4})^2 + \tfrac{25}{8}$$

So, the vertex is $(-\frac{1}{4}, \frac{25}{8})$ and since $x = \frac{1 \pm \sqrt{1 - 4(-2)(3)}}{-2 \cdot 2} = \frac{1 \pm \sqrt{25}}{-4} = \frac{1 \pm 5}{-4} = -\frac{6}{4}, 1$

thus its intercepts are $(-\frac{3}{2}, 0), (1, 0)$

Observation 3:

Let $f(x) = ax^2 + bx + c$, with $a \neq 0$, be a quadratic function. Show that the x-coordinate of the midpoint of the x-intercepts of f (whenever they exist!) is the x-coordinate of the vertex of f.

By quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\frac{1}{2}$

$$= \frac{(-b + \sqrt{b^2 - 4ac}) + (-b - \sqrt{b^2 - 4ac})}{2a} \cdot \frac{1}{2} = \frac{-2b}{2a} \cdot \frac{1}{2} = -\frac{b}{2a} = h.$$

Example 4: Find the maximum or minimum value of the function:

$f(t) = 100 - 49t - 7t^2$

Negative! $a = -7$, so maximum occurs at $x = -\frac{b}{2a} = -\frac{(-49)}{2(-7)} = \frac{49}{-14} = \frac{7}{-2}$

The maximum value is $f(-\frac{7}{2}) = 100 - 49(-\frac{7}{2}) - 7(-\frac{7}{2})^2$

$= 100 + \frac{343}{2} - \frac{343}{4} = \frac{400 + 686 - 343}{4} = \frac{743}{4}$

$g(x) = 100x^2 - 1500x$

positive! $a = 100$, so minimum occurs at $x = -\frac{b}{2a} = -\frac{-1500}{2 \cdot 100} = \frac{1500}{200} = \frac{15}{2}$.

The minimum value is $g(\frac{15}{2}) = 100(\frac{15}{2})^2 - 1500(\frac{15}{2})$

$= -5625$

Example 5:

Find a function of the form $f(x) = ax^2 + bx + c$ whose graph is a parabola with vertex $(1, -2)$ and that passes through the point $(4, 16)$.

For some d, $f(x) = d(x-1)^2 - 2$

and so $f(4) = 16 = d(4-1)^2 - 2 = d \cdot 3^2 - 2 = 9d - 2$

$\Rightarrow 18 = 9d \Rightarrow d = 2.$

Thus, $f(x) = 2(x-1)^2 - 2 = 2(x^2 - 2x + 1) - 2 = 2x^2 - 4x + 2 - 2 = 2x^2 - 4x.$

So, $f(x) = 2x^2 - 4x.$

Example 6 (Path of a Ball):

A ball is thrown across a playing field. Its path is given by the equation $y = -0.005x^2 + x + 5$, where x is the distance the ball has traveled horizontally, and y is its height above ground level, both measured in feet.

(a) What is the maximum height attained by the ball?

(b) How far has it traveled horizontally when it hits the ground?

ⓐ The maximum occurs at the vertex since $-0.005 < 0$.

That is, $k = \text{maximum height} = \dfrac{4ac - b^2}{4a} = \dfrac{4(-0.005)(5) - (1)^2}{4(-0.005)} = \dfrac{-0.1 - 1}{-0.02} = \dfrac{-1.1}{-0.02} = 55$ ft.

ⓑ This is the positive ~~nonzero~~ root of $-0.005x^2 + x + 5$, ~~evaluated~~

$X = \dfrac{-1 \pm \sqrt{1^2 - 4(-0.005)(5)}}{2(-0.005)} = \dfrac{-1 \pm \sqrt{1 + 0.1}}{-0.01} = \dfrac{-1 \pm \sqrt{1.1}}{-0.01} \Rightarrow X \approx -4.881,\ 204.881.$

Hence the ball traveled horizontally, approximately 204.881 feet.

Example 7 (Pharmaceuticals):

When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by $C(t) = 0.06t - 0.0002t^2$, where $0 \le t \le 240$ and the concentration is measured in mg/L. When is the maximum serum concentration reached, and what is that maximum concentration?

The maximum concentration is reached at the vertex:

$h = -\dfrac{b}{2a} = -\dfrac{0.06}{2(-0.0002)} = 150.$

Then, $k = 0.06(150) - 0.0002(150)^2 = 9 - 4.5 = 4.5.$

Thus the maximum concentration of 4.5 mg/L occurs at 150 minutes.