Today's Goal: We learn how two functions f and g can be combined to form new functions.

Assignments: **Homework (Sec. 3.6):** # 1,5,7,10,17,19,21,23,24,29,34,37,47 (pp. 275-277).

▶ **The Algebra of Functions:**

Let f and g be functions with domains A and B. We define new functions $f+g$, $f-g$, fg, and f/g as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{Domain } A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

Note: Consider the above definition $(f+g)(x) = f(x) + g(x)$. The $+$ on the left hand side stands for the operation of addition of functions. The $+$ on the right hand side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$. Similar remarks hold true for the other definitions.

Example 1: Let us consider the functions $f(x) = x^2 - 2x$ and $g(x) = 3x - 1$. Find $f+g$, $f-g$, fg, and f/g and their domains:

The domain of f is $(-\infty,\infty)$ and the domain of g is $(-\infty,\infty)$ as well.
Since $(-\infty,\infty) \cap (-\infty,\infty) = (-\infty,\infty)$, and $g(x)=0 \Rightarrow 0=3x-1 \Rightarrow \frac{1}{3}=x$,

	DOMAIN
$(f+g)(x) = (x^2 + 2x) + (3x-1) = x^2+x-1$	$(-\infty,\infty)$
$(f-g)(x) = (x^2-2x) - (3x-1) = x^2-5x+1$	$(-\infty,\infty)$
$(fg)(x) = (x^2-2x)(3x-1) = 3x^3-7x^2+2x$	$(-\infty,\infty)$
$(f/g)(x) = (x^2-2x)/(3x-1)$	$(-\infty, 1/3) \cup (1/3, \infty)$

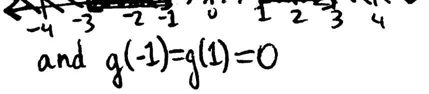
[1/3 is excluded because $g(1/3)=0$]

Example 2: Let us consider the functions $f(x) = \sqrt{9-x^2}$ and $g(x) = \sqrt{x^2-1}$. Find $f+g$, $f-g$, fg, and f/g and their domains:

$9-x^2 \geq 0 \Rightarrow -3 \leq x \leq 3$
so domain of f is
$[-3,3]$

$x^2-1 \geq 0 \Rightarrow x \geq 1$ or $x \leq -1$
so domain of g is
$(-\infty,-1] \cup [1,\infty)$

Now, domain of f
$\cap$ domain of g is
$[-3,3] \cap ((-\infty,-1] \cup [1,\infty))$
$= [-3,-1] \cup [1,3]$



and $g(-1)=g(1)=0$

	DOMAIN
$(f+g)(x) = \sqrt{9-x^2} + \sqrt{x^2-1}$	$[-3,-1] \cup [+1,+3]$
$(f-g)(x) = \sqrt{9-x^2} + \sqrt{x^2-1}$	$[-3,-1] \cup [1,3]$
$(fg)(x) = \sqrt{9-x^2} \cdot \sqrt{x^2-1} = \sqrt{(9-x^2)(x^2-1)}$	$[-3,-1] \cup [1,3]$
$(f/g)(x) = \sqrt{9-x^2} / \sqrt{x^2-1} = \sqrt{\left(\frac{9-x^2}{x^2-1}\right)}$	$[-3,-1) \cup (1,3]$

Now -1 and 1 are excluded, since $g(-1)=g(1)=0$

Example 5:

Let f and g be the functions considered in Example 3. Use the information provided by the graphs of f and g to find $f(g(1))$, $g(f(0))$, $f(g(0))$, and $g(f(4))$.

$g(1) = 0$ so $f(g(1)) = f(0) = 2$

$f(0) = 2$ so $g(f(0)) = g(2) = \frac{1}{2}$

$g(0) = -1$ so $f(g(0)) = f(-1) = \frac{5}{2}$

$f(4) = 0$ so $g(f(4)) = g(0) = -1$

Example 6:

Let $f(x) = \dfrac{x}{x+1}$ and $g(x) = 2x - 1$. Find the functions $f \circ g$, $g \circ f$, and $f \circ f$ and their domains.

Domain f: $x \neq -1$
Domain g: $\mathbb{R}$

$$f \circ g(x) = f(g(x)) = f(2x-1) = \frac{(2x-1)}{(2x-1)+1} = \frac{2x-1}{2x}$$

$$\underbrace{x}_{} \to \boxed{g} \to \underbrace{2x-1}_{} \to \boxed{f} \to \frac{2x-1}{2x}$$

$x \in \mathbb{R}$ AND $2x - 1 \neq -1$, $2x \neq 0$, $x \neq 0$ So $D_{f \circ g}$ = All real numbers except $x = 0$ $= (-\infty, 0) \cup (0, \infty)$

$$g \circ f(x) = g(f(x)) = g\left(\frac{x}{x+1}\right) = 2\left(\frac{x}{x+1}\right) - 1 = \frac{2x}{x+1} - 1 = \frac{2x}{x+1} - \frac{x+1}{x+1} = \frac{2x - \cancel{x} - 1}{x+1} = \frac{x-1}{x+1}$$

$$\underbrace{x}_{} \to \boxed{f} \to \underbrace{\frac{x}{x+1}}_{} \to \boxed{g} \to \frac{x-1}{x+1}$$

$x \neq -1$ AND $\frac{x}{x+1} \in \mathbb{R} \Rightarrow x \neq -1$ So $D_{g \circ f}$ = All real numbers except $x = -1$ $= (-\infty, -1) \cup (-1, \infty)$

$$f \circ f(x) = f(f(x)) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{2x+1}$$

$$\underbrace{x}_{} \to \boxed{f} \to \underbrace{\frac{x}{x+1}}_{} \to \boxed{f} \to \frac{x}{2x+1}$$

$x \neq -1$ AND $\frac{x}{x+1} \neq -1 \Rightarrow x \neq -x-1 \Rightarrow 2x \neq -1 \Rightarrow x \neq -\frac{1}{2}$

So $D_{f \circ f}$ = all real #'s except -1 and $-\frac{1}{2}$ $= (-\infty, -1) \cup (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

Example 7: Express the function $F(x) = \dfrac{x^2}{x^2+4}$ in the form $F(x) = f(g(x))$.

[There are many possible answers.]

If $f(x) = \frac{x}{x+4}$ and $g(x) = x^2$, then $f \circ g(x) = f(g(x)) = f(x^2) = \frac{(x^2)}{(x^2)+4} = F(x)$.

Example 8: Find functions f and g so that $f \circ g = H$ if $H(x) = \sqrt[3]{2 + \sqrt{x}}$.

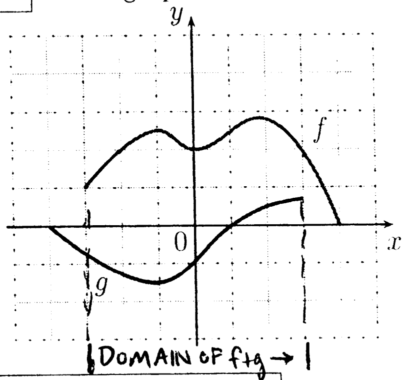
If $g(x) = \sqrt{x}$ and $f(x) = \sqrt[3]{2+x}$, then

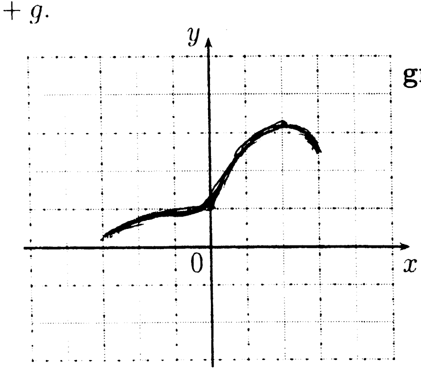
$f \circ g(x) = f(g(x)) = f(\sqrt{x}) = \sqrt[3]{2 + (\sqrt{x})} = H(x)$.

The graph of the function $f+g$ can be obtained from the graphs of f and g by **graphical addition**. This means that to obtain the value of $f+g$ at any point x we add the corresponding values of $f(x)$ and $g(x)$, that is, the corresponding y-coordinates.

Similar statements can be made for the other operations on functions.

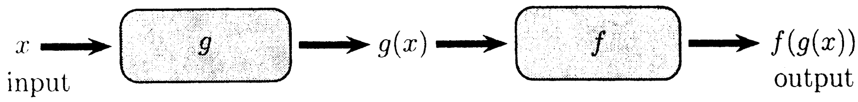
Example 3: Use graphical addition to sketch the graph of $f+g$.





graph of $f+g$

► **Composition of Functions:**



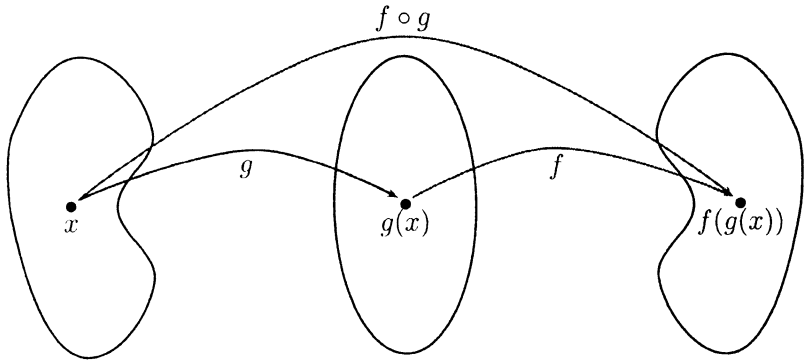
Machine diagram of $f \circ g$

Given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.

The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: 'f composed with g' or 'f after g')

$$\boxed{(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).}$$

WARNING: $f \circ g \neq g \circ f$.



Arrow diagram of $f \circ g$

Example 4: Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate:

$f(g(0)) = f(2) = 3(2) - 5 = 1$

$g(0) = 2 - 0^2 = 2$

$f(f(4)) = f(7) = 3(7) - 5 = 16$

$f(4) = 3(4) - 5 = 7$

$(f \circ g)(x) = f(g(x))$

$= f(2 - x^2)$

$= 3(2 - x^2) - 5$

$= -3x^2 + 1$

$g(f(0)) = g(-5) = 2 - (-5)^2 = -23$

$f(0) = 3(0) - 5 = -5$

$(g \circ g)(2) = g(g(2)) = g(-2) = 2 - (-2)^2 = -2$

$g(2) = 2 - (2)^2 = -2$

$(g \circ f)(x) = g(f(x))$

$= g(3x - 5)$

$= 2 - (3x - 5)^2$

$= -9x^2 + 30x - 23$