MA109, Activity 17: One-to-One Functions (Section 3.7, pp. 280-285) and Their Inverses

Date: ____________

Today's Goal: We first define one-to-one functions, which in turn allows us to introduce the notion of inverse of a one-to-one function. These topic will be of particular importance when we study exponential and logarithmic functions.

Assignments: **Homework (Sec. 3.7):** # 1,3,7,11,17,19,21,24,31,39,47,51,52 (pp. 286-289).

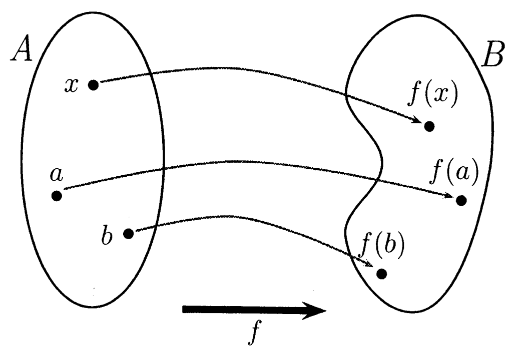
Definition of a One-One Function:

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

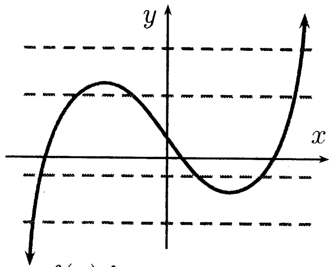
$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$



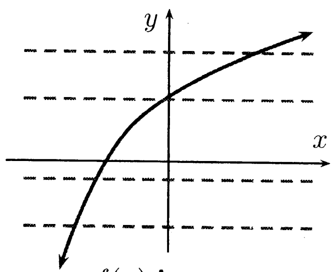
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test:

A function is one-to-one $\Longleftrightarrow$ no horizontal line intersects its graph more than once.



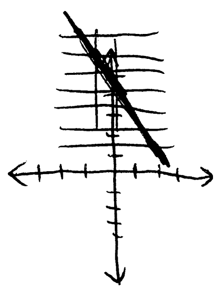
$f(x)$ **is not** one-to-one



$f(x)$ **is** one-to-one

Example 1: Show that the function $f(x) = 5 - 2x$ is one-to-one.

$y = 5-2x$ is a line with slope -2, so $y=f(x)$ passes the horizontal line test:

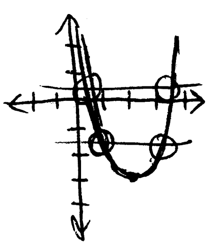


(Also if $5-2x = 5-2x'$, then $-2x = -2x'$ and $x = x'$ so f is one-to-one.)

Example 2:

Graph the function $f(x) = (x-2)^2 - 3$. The function is not one-to-one: Why? Can you restrict its domain so that the resulting function is one-to-one? (There is more than one correct answer.)

$f(x)$ is x^2 shifted down three units and to the right two units:



Every point other than the vertex lies on a horizontal line which intersects the graph at two different places, so f is not one-to-one.

If the domain is restricted to $[2, \infty)$ (or $(-\infty, 2]$), the function becomes one-to-one (because half of the parabola is chopped off).

Note: The domain can always be restricted to a single value so that no two elements will have the same image.

► **The Inverse of a Function:** One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function:

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$\boxed{f^{-1}(y) = x \iff f(x) = y,}$$

for any $y \in B$.

Example 3: Suppose $f(x)$ is a one-to-one function.

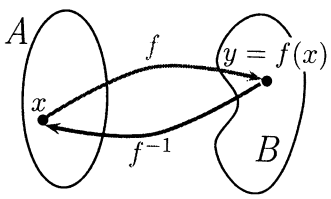
If $f(2) = 7,\ f(3) = -1,\ f(5) = 18,\ f^{-1}(2) = 6$ find:

$f^{-1}(7) = 2$ $\qquad$ $f(6) = 2$

$f^{-1}(-1) = 3$ $\qquad$ $f(f^{-1}(18)) = f(5) = 18$

If $g(x) = 9 - 3x$, then $g^{-1}(3) =$ 2

$9-3x=3 \Rightarrow -3x=-6 \Rightarrow x=2$



If f takes x to y, then f^{-1} takes y back to x. I.e., f^{-1} undoes what f does.

NOTE:

f^{-1} does NOT mean $\dfrac{1}{f}$.

Property of Inverse Functions:

Let $f(x)$ be a one-to-one function with domain A and range B.
The inverse function $f^{-1}(x)$ satisfies the following "cancellation" properties:

$\boxed{f^{-1}(f(x)) = x \text{ for every } x \in A}$ $\qquad$ $\boxed{f(f^{-1}(x)) = x \text{ for every } x \in B}$

Conversely, any function $f^{-1}(x)$ satisfying the above conditions is the inverse of $f(x)$.

Example 4: Show that the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverses of each other.

$f(g(x)) = f(x^{1/5}) = (x^{1/5})^5 = x^1 = x$

and

$g(f(x)) = g(x^5) = (x^5)^{1/5} = x^1 = x$

Example 5: Show that the functions $f(x) = \dfrac{1+3x}{5-2x}$ and $g(x) = \dfrac{5x-1}{2x+3}$ are inverses of each other.

$$f(g(x)) = f\left(\frac{5x-1}{2x+3}\right) = \frac{1+3\left(\frac{5x-1}{2x+3}\right)}{5-2\left(\frac{5x-1}{2x+3}\right)} = \frac{(2x+3)+3(5x-1)}{5(2x+3)-2(5x-1)}$$

$$= \frac{2x+3+15x-3}{10x+15-10x+2} = \frac{17x}{17} = x$$

$$g(f(x)) = g\left(\frac{1+3x}{5-2x}\right) = \frac{5\left(\frac{1+3x}{5-2x}\right)-1}{2\left(\frac{1+3x}{5-2x}\right)+3} = \frac{5(1+3x)-1(5-2x)}{2(1+3x)+3(5-2x)}$$

$$= \frac{5+15x-5+2x}{2+6x+15-6x} = \frac{17x}{17} = x$$

How to find the Inverse of a One-to-One Function:

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 6: Find the inverse of $y = 4x - 7$.

$y = 4x-7$

$y+7 = 4x$

$\dfrac{y+7}{4} = x$

$y = \dfrac{x+7}{4}$

Example 7: Find the inverse of $y = \dfrac{1}{x+2}$.

$$y = \frac{1}{x+2}$$

$$(x+2)y = 1$$

$$x+2 = \frac{1}{y}$$

$$x = \frac{1}{y} - 2 \quad \longrightarrow \quad y = \frac{1}{x} - 2$$

Example 8: Find the inverse of $y = \dfrac{2-x}{x+2}$.

$$y = \frac{2-x}{x+2}$$

$$(x+2)y = 2-x$$

$$xy + 2y = 2 - x$$

get all x's on one side, and everything else on the other

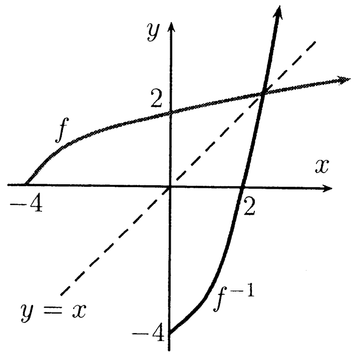
$$xy + x = 2 - 2y$$

factor out x

$$x(y+1) = 2-2y$$

$$x = \frac{2-2y}{y+1} \quad \longrightarrow \quad y = \frac{2-2x}{x+1}$$

Graph of the Inverse Function: The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**



The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4} \qquad \text{and} \qquad f^{-1}(x) = x^2 - 4,\ x \geq 0.$$

Example 9: Find the inverse of the function $f(x) = 1 + \sqrt{1+x}$.
Find the domain and range of f and f^{-1}. Graph f and f^{-1} on the same cartesian plane.

$f(x) = 1 + \sqrt{1+x}$ has a domain of $[-1, \infty)$ because $1 + x \geq 0 \Rightarrow x \geq -1$.

$f(x) = 1 + \sqrt{1+x}$ has a range of $[1, \infty)$ because $\sqrt{1+x} \geq 0$ so $1 + \sqrt{1+x} \geq 1$.

Thus, the domain of f^{-1} is $[1, \infty)$, and the range of f^{-1} is $[-1, \infty)$.