

**Today's Goal:** Functions defined by polynomial expressions are called polynomial functions. The graphs of polynomial functions are beautiful, smooth curves that can increase and decrease several times. For this reason they are useful in modeling many real-world situations.

**Assignments:** **Homework (Sec. 4.1):** # 1,5,11,15,18,23,27,29,39,51,52 (pp. 322-325).

► **Polynomial Functions:** A polynomial function of degree  $n$  is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where  $n$  is a non-negative integer and  $a_n \neq 0$ . The numbers  $a_0, a_1, \dots, a_n$  are called the coefficients of the polynomial. The number  $a_0$  is the **constant coefficient** or **constant term**. The number  $a_n$ , the coefficient of the highest power, is the **leading coefficient**, and the term  $a_n x^n$  is the **leading term**.

► **Graphs of Polynomials:**

The graphs of polynomials of degree 0 and 1 are lines; the graphs of polynomials of degree 2 are parabolas. The greater the degree of the polynomial, the more complicated the graph can be. However, **the graph of a polynomial function is always a smooth curve; that is, it has no breaks or sharp corners.**

► **End Behavior and the Leading Term:**

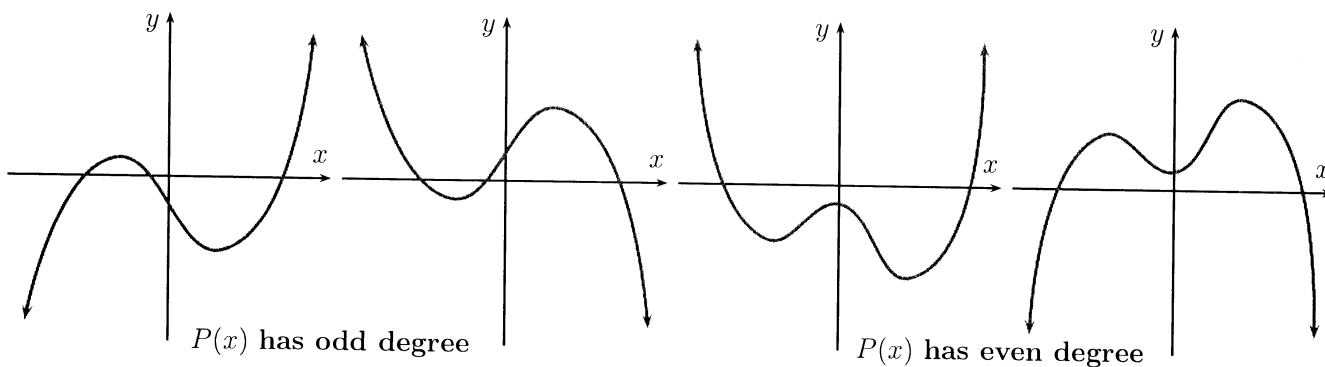
The polynomial  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  has the same end behavior as the monomial  $Q(x) = a_n x^n$ , so its end behavior is determined by the degree  $n$  and the sign of the leading coefficient  $a_n$ .

IF  $y = P(x)$  has **odd degree** and  $a_n$  is **positive**, then  $y \rightarrow +\infty$  as  $x \rightarrow +\infty$  and  $y \rightarrow -\infty$  as  $x \rightarrow -\infty$ .

IF  $y = P(x)$  has **odd degree** and  $a_n$  is **negative**, then  $y \rightarrow +\infty$  as  $x \rightarrow -\infty$  and  $y \rightarrow -\infty$  as  $x \rightarrow +\infty$ .

IF  $y = P(x)$  has **even degree** and  $a_n$  is **positive**, then  $y \rightarrow +\infty$  as  $x \rightarrow \pm\infty$ .

IF  $y = P(x)$  has **even degree** and  $a_n$  is **negative**, then  $y \rightarrow -\infty$  as  $x \rightarrow \pm\infty$ .



► **Using Zeros to Graph Polynomials:**

**Real Zeros of Polynomials:** If  $P(x)$  is a polynomial and  $c \in \mathbb{R}$ , then the following are equivalent:

1.  $c$  is a zero of  $P(x)$ ;
2.  $x = c$  is a solution of  $P(x) = 0$ ;
3.  $x - c$  is a factor of  $P(x)$ ;
4.  $x = c$  is an  $x$ -intercept of the graph of  $y = P(x)$ .

**Example 1:** Determine the end behavior of the polynomial  $P(x) = 3(x^2 - 4)(x - 1)^3$ .

$$3 \cdot x^2 \cdot x \cdot x \cdot x = 3x^5, \quad \text{degree} = 5 \quad \& \quad a_n = 3 > 0$$

End Behavior:  $y \rightarrow \infty$  as  $x \rightarrow \infty$  &  $y \rightarrow -\infty$  as  $x \rightarrow -\infty$

### Intermediate Value Theorem for Polynomials:

If  $P(x)$  is a polynomial function and  $P(a)$  and  $P(b)$  have opposite signs, then there exists at least one value  $c$  between  $a$  and  $b$  for which  $P(c) = 0$ .

**Example 2:** Use the Intermediate Value Theorem to show that the polynomial  $P(x) = -x^4 + 3x^3 - 2x + 1$  has a zero in the interval  $[2, 3]$ .

$$P(2) = -(2)^4 + 3(2)^3 - 2(2) + 1 = -16 + 24 - 4 + 1 = 5$$

$$P(3) = -(3)^4 + 3(3)^3 - 2(3) + 1 = -81 + 81 - 6 + 1 = -5$$

By IVT, there is a  $c$  in  $[2, 3]$  so that  $P(c) = 0$

### Guidelines for Graphing Polynomial Functions:

- Zeros:** Factor the polynomial to find all its real zeros; these are the  $x$ -intercepts of the graph.
- Test Points:** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the  $x$ -axis on the intervals determined by the zeros. Include the  $y$ -intercept in the table.
- End Behavior:** Determine the end behavior of the polynomial.
- Graph:** Plot the intercepts and the other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

**Example 3:** Let  $P(x) = (x+2)(x-1)(x-3)$ . Find the zeros of  $P(x)$  and sketch its graph.

Zeros:  $x = -2, x = 1, \text{ and } x = 3$

Test Points

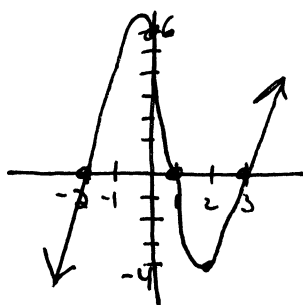
$$P(0) = 6$$

$$P(2) = -4$$

$$\text{deg } P = 3$$

$$a_n = 1 > 0$$

$$\text{end behavior: } y \rightarrow \infty \text{ as } x \rightarrow \infty \\ \text{and } y \rightarrow -\infty \text{ as } x \rightarrow -\infty$$



**Example 4:** Let  $P(x) = -2x^4 - x^3 + 3x^2$ . Find the zeros of  $P(x)$  and sketch its graph.

Zeros

$$0 = -2x^4 - x^3 + 3x^2$$

$$= x^2(-2x^2 - x + 3)$$

$$= x^2(-2x-3)(x-1)$$

$$\text{Zeros at } x = 0, x = -\frac{3}{2}, \text{ and } x = 1$$

$$\text{deg } P = 4$$

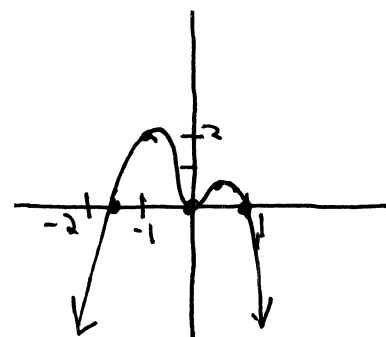
$$a_n = -2 < 0$$

$$\text{end behavior: } y \rightarrow -\infty \text{ as } x \rightarrow \pm\infty$$

Test Points

$$P(-1) = 2$$

$$P\left(\frac{1}{2}\right) = \frac{1}{2}$$

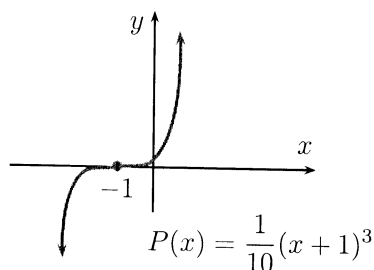
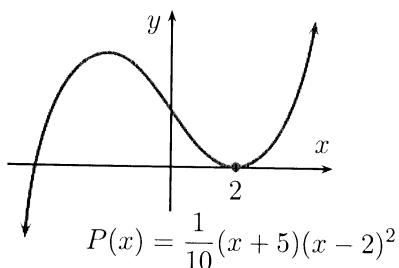


► **Shape of the Graph Near a Zero:**

If the factor  $(x - c)$  appears  $m$  times in the complete factorization of  $P(x)$ , i.e.,  $P(x) = (x - c)^m \cdot Q(x)$  with  $Q(c) \neq 0$ , then  $c$  is said to be a zero of **multiplicity**  $m$ .

If  $c$  is a zero of **even** multiplicity, then the graph of  $y = P(x)$  **touches** the  $x$ -axis at  $(c, 0)$ .

If  $c$  is a zero of **odd** multiplicity, then the graph of  $y = P(x)$  **crosses** the  $x$ -axis at  $(c, 0)$ .



**Example 5:** Let  $f(x) = x^3(x^2 + 1)(x - 3)^4$ . List each real zero and its multiplicity. Determine if the graph crosses or touches the  $x$ -axis. Find the degree and the end behavior of  $f(x)$ . Find the  $y$ -intercept of  $y = f(x)$ .

Zeros	Multiplicity	
$x = 0$	3	crosses
$x = 3$	4	touches

$$\begin{aligned} \text{deg } f &= \text{deg}(x^3 \cdot x^2 \cdot x \cdot x \cdot x \cdot x) \\ &= \text{deg}(x^9) \\ &= 9 \end{aligned}$$

$y$ -intercept

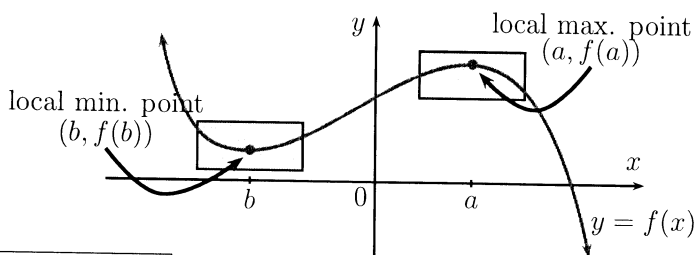
Let  $x = 0$

$$f(0) = 0^3(0^2 + 1)(0 - 3)^4 = 0$$

$(0, 0)$

► **Local Maxima and Minima of Polynomials:**

If a point  $(a, f(a))$  is the highest point on the graph of  $f$  within some viewing rectangle, then  $f(a)$  is a **local maximum value** of  $f$ . If a point  $(b, f(b))$  is the lowest point on the graph of  $f$  within some viewing rectangle, then  $f(b)$  is a **local minimum value** of  $f$ .



**Local Extrema of Polynomials:**

If  $P(x)$  is a polynomial of degree  $n$ , then the graph of  $P(x)$  has at most  $n - 1$  local extrema values.

**Example 6:** Sketch the family of polynomials  $y = x^3 - cx^2$  for  $c = 0, 1, 2,$  and  $3$ . How does the change in the value of  $c$  affect the graph?

For each value of  $c$   
 We have that the end behavior is  
 $y \rightarrow \infty$  as  $x \rightarrow \infty$   
 $y \rightarrow -\infty$  as  $x \rightarrow -\infty$   
 since the degree is 3 and  $a_n = 1 > 0$

Zeros

$$\begin{aligned} 0 &= x^3 - cx^2 \\ &= x^2(x - c) \end{aligned}$$

$x = 0$  multiplicity 2  
 $x = c$  multiplicity 1

