MA109, Activity 19: Dividing Polynomials (Section 4.2, pp. 325-331) Date: ________

Today's Goal:	So far we have been studying polynomial functions *graphically*. We now begin to study polynomials *algebraically*. Most of our work will be concerned with factoring polynomials, and to factor, we need to know how to divide polynomials.
Assignments:	**Homework (Sec. 4.2):** # 1,3,5,11,13,19,22,27,31,36,43,53 (pp. 331-332).

Given the integers 23 and 5 we can 'divide' one by the other. We obtain: $\frac{23}{5} = 4 + \frac{3}{5}$ or $23 = 4 \cdot 5 + 3$. In general, if a and b are non-zero integers, then there exist unique integers q and r such that

$$a = q \cdot b + r \qquad \text{and} \qquad 0 \le r < |b|,$$

where q is the *quotient* and r the *remainder*. This is the usual '*long division*' familiar from elementary arithmetic.

Example 1: Divide 63 by 12.

$\frac{63}{12} = 5 + \frac{3}{12}$ or $63 = 5 \cdot 12 + 3$

▶ **Long Division of Polynomials:** Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials.*

Division Algorithm: If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree strictly less than the degree of $D(x)$, such that

$$P(x) = Q(x) \cdot D(x) + R(x)$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively; $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

Example 2: Divide the polynomial

$$P(x) = 2x^2 - x - 4 \quad \text{by} \quad D(x) = x - 3.$$

$$\begin{array}{r|l} & 2x + 5 \\ \hline x - 3 & 2x^2 - x - 4 \\ & 2x^2 - 6x \\ \hline & + 5x - 4 \\ & -(5x - 15) \\ \hline & +11 \end{array}$$

$2x^2 - x - 4 = (2x+5)(x-3) + 11$

✓: $(2x+5)(x-3) + 11 =$

$2x^2 - 6x + 5x - 15 + 11 =$

$2x^2 - x - 4$ ✓

(Complete the above table and check your work!)

Example 3: Divide the polynomial

$$P(x) = x^4 - x^3 + 4x + 2 \quad \text{by} \quad D(x) = x^2 + 3.$$

$$\begin{array}{r|l} & x^2 - x - 3 \\ \hline x^2 + 0x + 3 & x^4 - x^3 + 0x^2 + 4x + 2 \\ & -(x^4 + 0x^3 + 3x^2) \\ \hline & -x^3 - 3x^2 + 4x \\ & -(-x^3 - 0x^2 - 3x) \\ \hline & -3x^2 + 7x + 2 \\ & (-3x^2 - 0x - 9) \\ \hline & 7x + 11 \end{array}$$

$x^4 - x^3 + 4x + 2 = (x^2 - x - 3)(x^2 + 3) + (7x + 11)$

✓: $(x^2 - x - 3)(x^2 + 3) + 7x + 11 =$

$x^4 - x^3 - 3x^2 + 3x^2 - 3x - 9 + 7x + 11 =$

$x^4 - x^3 + 4x + 2$ ✓

Example 4:

Find the quotient $Q(x)$ and the remainder $R(x)$ when $f(x) = 3x^3 + 2x^2 - x + 3$ is divided by $g(x) = x - 4$.

$$3x^2 + 14x + 55 \leftarrow Q(x)$$

$$x - 4 \overline{\big) 3x^3 + 2x^2 - x + 3}$$

$$-(3x^3 - 12x^2)$$

$$+14x^2 - x + 3$$

$$-(14x^2 - 56x)$$

$$55x + 3$$

$$-(55x - 220)$$

$$223 \leftarrow R(x)$$

$\checkmark$: $(3x^2 + 14x + 55)(x - 4) + 223 =$

$3x^3 + 14x^2 + 55x - 12x^2 - 56x - 220 + 223$

$= 3x^3 + 2x^2 - x + 3$ $\checkmark$

Example 5:

Find the quotient $Q(x)$ and the remainder $R(x)$ when $f(x) = x^5 - 4x^3 + x$ is divided by $g(x) = x + 3$.

$$x^4 - 3x^3 + 5x^2 - 15x + 46 \leftarrow Q(x)$$

$$x + 3 \overline{\big) x^5 + 0x^4 - 4x^3 + 0x^2 + x + 0}$$

$$-(x^5 + 3x^4)$$

$$-3x^4 - 4x^3$$

$$-(-3x^4 - 9x^3)$$

$$5x^3 + 0x^2$$

$$-(5x^3 + 15x^2)$$

$$-15x^2 + x$$

$$-(-15x^2 - 45x)$$

$$46x + 0$$

$$-(46x + 138)$$

$$-138 \leftarrow R(x).$$

▶ **The Remainder and Factor Theorems:**

Next, we see how synthetic division can be used to evaluate polynomials 'easily.'

> **Remainder Theorem:**
> If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Proof: If the divisor $D(x)$ is of the form $x - c$, then the remainder MUST be a constant R. Thus

$$P(x) = Q(x) \cdot (x - c) + R.$$

Setting $x = c$ in the above equation gives that $P(c) = Q(c) \cdot 0 + R = R$. Thus

$$\boxed{P(x) = Q(x) \cdot (x - c) + P(c).}$$

From the boxed equation we obtain our next theorem, which says that the *zeros* of a polynomial correspond to the *linear factors* of the polynomial.

> **Factor Theorem:**
> The number c is a zero of $P(x)$ if and only if $x - c$ is a factor of $P(x)$;
> that is, $P(x) = Q(x) \cdot (x - c)$ for some polynomial $Q(x)$.

Example 6: Let $P(x) = x^3 + 2x^2 - 7$.

(a) Find the quotient and the remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$.

(a)

$$x^2$$

$$x + 2 \overline{\big) x^3 + 2x^2 + 0x - 7}$$

$$-(x^3 + 2x^2)$$

$$-7$$

So $\underbrace{x^3 + 2x^2 - 7}_{P(x)} = \underbrace{x^2}_{Q(x)}\underbrace{(x+2)}_{(x-(-2))} \underbrace{- 7}_{P(-2)}$

(b) By the Remainder Thm., $P(-2) = -7$

$\checkmark$: $P(-2) = (-2)^3 + 2(-2)^2 - 7 = -8 + 8 - 7 = -7$ $\checkmark$.

Example 7: Use the Factor Theorem to determine whether $x+2$ is a factor of $f(x) = 3x^6 + 2x^3 - 176$.

According to the Factor Theorem, $x+2$ is a factor of $f(x)$ if and only if -2 is a zero of $f(x)$, i.e. $f(-2)=0$.

$$
\begin{aligned}
f(-2) &= 3(-2)^6 + 2(-2)^3 - 176 \\
&= 3(+64) + 2(-8) - 176 \\
&= 192 - 16 - 176 \\
&= 192 - 192 = 0
\end{aligned}
$$

So $x+2$ is a factor of $f(x) = 3x^6 + 2x^3 - 176$!

Example 8:

Find a polynomial of degree 3 that has zeros 1, -2, and 3, and in which the coefficient of x^2 is 3.

If your polynomial has zeros $1, -2$ and 3, it has factors $(x-1)$, $(x-(-2))$ and $(x-3)$. Notice

$$
\begin{aligned}
(x-1)(x+2)(x-3) &= (x^2+x-2)(x-3) \\
&= x^3 + x^2 - 2x - 3x^2 - 3x + 6 \\
&= x^3 - 2x^2 - 5x + 6
\end{aligned}
$$

Now if we want the coefficient of x^2 to be 3, we need to multipl

$(-2)(-\frac{3}{2}) = 3$.

$$
\begin{aligned}
\left(-\tfrac{3}{2}\right)(x-1)(x+2)(x-3) &= -\tfrac{3}{2}(x^3 - 2x^2 - 5x + 6) \\
&= -\tfrac{3}{2}x^3 + 3x^2 + \tfrac{15}{2}x - 9.
\end{aligned}
$$

Example 9: Let $P(x) = 2x^3 + 3x^2 - 17x - 30$.

- Is 3 a zero of $P(x)$? What does this tell you about the factors of $P(x)$? What does it tell you about the graph of $y = P(x)$?
- Is 2 a zero of $P(x)$? What does this tell you about the factors of $P(x)$? What does it tell you about the graph of $y = P(x)$?

- Is 3 a zero of $P(x)$? $P(3) = 2(3)^3 + 3(3)^2 - 17(3) - 30 = 54 + 27 - 51 - 30 = 0$

  Yes, 3 is a zero of $P(x)$. This means $(x-3)$ is a factor of $P(x)$ and the graph of $y = P(x)$ touches or crosses the x-axis at $x = 3$.

- Is 2 a zero of $P(x)$? $P(2) = 2(2)^3 + 3(2)^2 - 17(2) - 30 = 16 + 12 - 34 - 30 = 28 - 64 = -36 \neq 0$.

  No, 2 is <u>not</u> a zero of $P(x)$.

  So $(x-2)$ is <u>not</u> a factor of $P(x)$ and the graph of $y = P(x)$ does not touch or cross the x-axis at $x = 2$.

Example 10:

The graph of a polynomial has x-intercepts at $(2,0)$ and $(-5,0)$. What does this tell you about the polynomial?

This means $x=2$ and $x=-5$ are zeros of this polynomial, which, in turn, means $(x-2)$ and $(x-(-5))$ are factors of the polynomial.

Remember $x-(-5) = x+5$.