MA109, Activity 20: Real Zeros of Polynomials (Section 4.3, pp. 333-340) Date: ________

Today's Goal:	We study some algebraic methods that help us find the real zeros of a polynomial, and thereby factor the polynomial.
Assignments:	**Homework (Sec. 4.3):** # 1,5,7,9,11,19,23,39,41,49,57,61,94 (pp. 341-344).

▶ **Rational Zeros of Polynomials:** Consider the polynomial

$$P(x) = (x+2)(2-x)(x-3) = -x^3 + 3x^2 + 4x - 12.$$

From the factored form we see that the zeros of $P(x)$ are -2, 2, and 3. From the expanded form we see that the constant term -12 is obtained by multiplying $2 \cdot 2 \cdot (-3)$. This means that the zeros of the polynomial are all factors of the constant term. This observation can be generalized as follows.

Rational Zeros Theorem: If the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then every rational zero of $P(x)$ is of the form

$$\frac{p}{q},$$

where p is a factor of the constant coefficient a_0 and q is a factor of the leading coefficient a_n.

Example 1: List all possible rational zeros of $P(x) = 2x^4 - x^2 - 7$.

Every RATIONAL zero is of the form $\frac{p}{q}$ where p divides -7 AND q divides 2. So $\frac{p}{q} \in \{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{7}{1}, \pm\frac{7}{2}\}$.

Finding the Rational Zeros of a Polynomial:

1. **List Possible Zeros:** List all the possible rational zeros using the Rational Zeros Theorem.
2. **Find a Zero, then Divide:** Evaluate the polynomial at each of the candidates for a rational zero until you find a zero, c. Then divide the polynomial by $x - c$ and note the quotient.
3. **Repeat:** Repeat Steps **1.** and **2.** for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

Example 2:

Find the real zeros of $f(x) = 2x^3 - 5x^2 - 4x + 3$. Write $f(x)$ in factored form and sketch its graph.

THE POSSIBLE RATIONAL ZEROS OF f ARE $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$.

$f(1) = 2(1)^3 - 5(1)^2 - 4(1) + 3 = -4 \neq 0$.

$f(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3 = 0 \rightarrow$ So $(x - (-1))$ IS A ROOT OF $f(x)$.

USING LONG DIVISION

$$
\begin{array}{r}
2x^2 - 7x + 3 \\
x+1 \,\overline{)\, 2x^3 - 5x^2 - 4x + 3} \\
-(2x^3 + 2x^2) \\
\hline
-7x^2 - 4x \\
-(-7x^2 - 7x) \\
\hline
3x + 3 \\
-(3x + 3) \\
\hline
0
\end{array}
$$

USING SYNTHETIC DIVISION

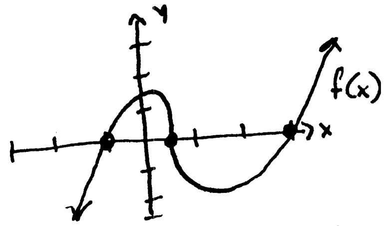
-1	2	-5	-4	3
		-2	7	-3
	2	-7	3	0

Your book teaches this, but we are not using it in this class. You do not need to know this

IN EITHER CASE WE SEE $f(x) = (2x^2 - 7x + 3)(x + 1)$,

AND WE USE THE QUADRATIC formula TO FIND THE zeros ~~roots~~ OF $2x^2 - 7x + 3$, WHICH ARE $x = 3$ AND $x = \frac{1}{2}$.

So ~~THE~~ $f(x) = (x+1)(x-3)(2x-1)$.



* NOTE: Degree is ODD w/ positive leading coefficient.

Example 3: Find the real solutions of the equation $x^4 - 2x^3 - 6x^2 + 7x + 6 = 0$.

The possible rational roots of $f(x) = x^4 - 2x^3 - 6x^2 + 7x + 6$ Are $\left\{\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}\right\}$

Notice THAT $f(3) = 3^4 - 2(3)^3 - 6(3)^2 + 7(3) + 6 = 0$, So $(x-3)$ IS A factor of $f(x)$ (AND $x = 3$ IS A solution TO THE EQUATION)

$$
\begin{array}{r}
x^3 + x^2 - 3x - 2 \\
x-3 \,\overline{)\, x^4 - 2x^3 - 6x^2 + 7x + 6} \\
-(x^4 - 3x^3) \\
\hline
x^3 - 6x^2 \\
-(x^3 - 3x^2) \\
\hline
-3x^2 + 7x \\
-(-3x^2 + 9x) \\
\hline
-2x + 6 \\
-(-2x + 6) \\
\hline
0
\end{array}
$$

So $f(x) = (x-3)(x^3 + x^2 - 3x - 2)$

Now THE possible rational roots of $x^3 + x^2 - 3x - 2$ Are $\left\{\pm\frac{1}{1}, \pm\frac{2}{1}\right\}$ AND THE ONLY ONE of these that IS A zero of THIS function is -2, SINCE $f(-2) = (-2)^3 + (-2)^2 - 3(-2) - 2 = 0$.

So WE DIVIDE AGAIN AND get $f(x) = (x-3)(x+2)(x^2 - x - 1)$. Finally, THE QUADRATIC formula gives $x = \frac{1 \pm \sqrt{5}}{2}$ AS roots of $x^2 - x - 1$. So THE solutions to $f(x) = 0$ ARE $\left\{3, -2, \frac{1 \pm \sqrt{5}}{2}\right\}$.