MA109, Activity 21: Rational Expressions (Section P.7, pp. 50-57) **Date:** ________

Today's Goal:	In this activity we learn how to perform algebraic operations on fractional (or rational) expressions.
Assignments:	**Homework (Sec. P.7):** # 3, 6, 11, 16, 19, 25, 30, 31, 37, 47, 52, 57, 65, 78, 81 (pp. 58-60).

A quotient of two algebraic expressions is called a "fractional expression." For example,

$$\frac{3+\sqrt{yz}}{x^2+y-1} \quad \text{or} \quad \frac{5x^2+4x-1}{\sqrt{x+1}}.$$

A "rational expression" is a fractional expression where both the numerator and denominator are polynomials:

$$\frac{1-3x}{x^2+2} \quad \text{or} \quad \frac{x^4+2x-1}{6x^5-4x^4+3}.$$

▶ The Domain of an Algebraic Expression:

In general, an algebraic expression may not be defined for all values of the variable. The "domain" of an algebraic expression is the set of real numbers that the variable is permitted to have.

Example 1: Find the domains of the following expressions:

- $3x^2-2x+\pi$

  $(-\infty,\infty)$

- $\sqrt{1-x}$

  $1-x\geq 0 \Rightarrow 1\geq x$

  Domain is $(-\infty,1]$.

- $\dfrac{2x-1}{x^2+x-6}$

  $x^2+x-6\neq 0 \Rightarrow (x-2)(x+3)\neq 0$

  $\Rightarrow x-2\neq 0$ and $x+3\neq 0$

  $\Rightarrow x\neq 2$ and $x\neq -3$

  Domain is $(-\infty,-3)\cup(-3,2)\cup(2,\infty)$.

- $\dfrac{1}{(x-5)\sqrt{x-2}}$

  $(x-5)\sqrt{x-2}\neq 0$ & $x-2\geq 0 \Rightarrow$

  $x-5\neq 0$ & $\sqrt{x-2}\neq 0$ & $x-2\geq 0 \Rightarrow$

  $x-5\neq 0$ & $x-2>0 \Rightarrow$

  $x\neq 5$ & $x>2$

  Domain is $(2,5)\cup(5,\infty)$.

▶ Simplifying Fractional Expressions:

To simplify fractional expressions, we factor both numerator and denominator and use the following property of fractions:

$$\boxed{\frac{AC}{BC}=\frac{A}{B}}$$

This allows us to cancel common factors from the numerator and denominator.

Example 2: Perform the indicated operations and simplify:

- $\dfrac{x^2+6x+8}{x^2+5x+4} = \dfrac{(x+2)(x+4)}{(x+1)(x+4)} = \dfrac{x+2}{x+1}$

- $\dfrac{1-x^2}{x^3-1} = \dfrac{(1-x)(1+x)}{(x-1)(x^2+x+1)} = \dfrac{-(x-1)(1+x)}{(x-1)(x^2+x+1)} = \dfrac{-(1+x)}{x^2+x+1}$

▶ **Multiplying and Dividing Fractional Expressions:**

To multiply or divide fractional expressions, we use the following properties of fractions:

$$\boxed{\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}} \qquad \boxed{\frac{A/B}{C/D}=\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}=\frac{AD}{BC}}$$

Example 3: Perform the indicated operations and simplify:

- $\frac{x^2-x-12}{x^2-9}\cdot\frac{3+x}{4-x} = \frac{(x^2-x-12)(3+x)}{(x^2-9)(4-x)}$
  $= \frac{(x-4)(x+3)(3+x)}{-(x+3)(x-3)(x-4)}$
  $= \frac{3+x}{3-x}$

- $\frac{x}{y/z} = \frac{x}{1}\cdot\frac{z}{y} = \frac{xz}{y}$

- $\frac{\frac{2x^2-3x-2}{x^2-1}}{\frac{2x^2+5x+2}{x^2+x-2}} = \frac{2x^2-3x-2}{x^2-1}\cdot\frac{x^2+x-2}{2x^2+5x+2}$
  $= \frac{(x-2)(2x+1)}{(x+1)(x-1)}\cdot\frac{(x-1)\cancel{(x+2)}}{\cancel{(x+2)}(2x+1)}$
  $= \frac{x-2}{x+1}$

▶ **Adding and Subtracting Fractional Expressions:**

To add or subtract fractional expressions, we first find the least common denominator and then use the following property of fractions:

$$\boxed{\frac{A}{C}\pm\frac{B}{C}=\frac{A\pm B}{C}}$$

Example 4: Perform the indicated operations and simplify:

- $u+1+\frac{u}{1+u} = \frac{(u+1)^2+u}{1+u}$
  $= \frac{u^2+3u+1}{1+u}$

- $\frac{x}{(x+1)^2}+\frac{2}{x+1} = \frac{x+2(x+1)}{(x+1)^2}$
  $= \frac{3x+2}{(x+1)^2}$

- $\frac{1}{x^2}+\frac{1}{x^2+x} = \frac{(x^2+x)+x^2}{x^2(x^2+x)}$
  $= \frac{2x^2+x}{x^4+x^3}$
  $= \frac{2x+1}{x^3+x^2}$

- $\frac{2}{a^2}-\frac{3}{ab}+\frac{4}{b^2} = \frac{2b^2}{a^2b^2}-\frac{3ab}{a^2b^2}+\frac{4a^2}{a^2b^2}$
  $= \frac{2b^2-3ab+4a^2}{a^2b^2}$

► **Compound Fractions:** A compound fraction is a fraction in which the numerator, the denominator, or both, are themselves fractional expressions.

Example 5: Simplify the following compound fractional expressions:

- $\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{x^2}-\frac{1}{y^2}} = \dfrac{\frac{x^2-y^2}{xy}}{\frac{y^2-x^2}{x^2y^2}} =$

$= \dfrac{x^2-y^2}{xy} \cdot \dfrac{x^2y^2}{-(x^2-y^2)} = -xy$

- $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}} = \dfrac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}} = \dfrac{\frac{y^2-x^2}{x^2y^2}}{\frac{y+x}{xy}} =$

$= \dfrac{y^2-x^2}{x^2y^2} \cdot \dfrac{xy}{y+x} = \dfrac{(y+x)(y-x)}{xy(y+x)} = \dfrac{y-x}{xy}$

- $1+\dfrac{1}{1+\frac{1}{1+x}} = 1+\dfrac{1}{\frac{1+x+1}{1+x}} =$

$= 1+\dfrac{1+x}{x+2} = \dfrac{x+2+1+x}{x+2} =$

$= \dfrac{2x+3}{x+2}$

- $\dfrac{\left(a+\frac{1}{b}\right)^m\left(a-\frac{1}{b}\right)^n}{\left(b+\frac{1}{a}\right)^m\left(b-\frac{1}{a}\right)^n} = \dfrac{\left(\frac{ab+1}{b}\right)^m\left(\frac{ab-1}{b}\right)^n}{\left(\frac{ab+1}{a}\right)^m\left(\frac{ab-1}{a}\right)^n}$

$= \dfrac{\frac{(ab+1)^m}{b^m}\cdot\frac{(ab-1)^n}{b^n}}{\frac{(ab+1)^m}{a^m}\cdot\frac{(ab-1)^n}{a^n}} = \dfrac{(ab+1)^m(ab-1)^n}{b^m b^n}\cdot\dfrac{a^m a^n}{(ab+1)^m(ab-1)^n} =$

$= \dfrac{a^m a^n}{b^m b^n} = \dfrac{a^{m+n}}{b^{m+n}}$

► **Rationalizing the Denominator or the Numerator:**

If a fraction has a denominator (or a numerator) of the form $A\sqrt{B}+C\sqrt{D}$, we may rationalize it by multiplying numerator and denominator by the conjugate radical $A\sqrt{B}-C\sqrt{D}$. Observe that:

$$(A\sqrt{B}+C\sqrt{D})(A\sqrt{B}-C\sqrt{D}) = A^2B-C^2D.$$

Example 6: Rationalize the denominator or the numerator of the following fractional expressions:

- $\dfrac{1}{2-\sqrt{3}} = \dfrac{1}{2-\sqrt{3}}\cdot\dfrac{2+\sqrt{3}}{2+\sqrt{3}}$

$= \dfrac{2+\sqrt{3}}{4-3}$

$= 2+\sqrt{3}$

- $\dfrac{\sqrt{3}+\sqrt{5}}{2} = \dfrac{\sqrt{3}+\sqrt{5}}{2}\cdot\dfrac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}}$

$= \dfrac{3-5}{2(\sqrt{3}-\sqrt{5})} = \dfrac{-2}{2(\sqrt{3}-\sqrt{5})} =$

$= \dfrac{-1}{\sqrt{3}-\sqrt{5}}$

- $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}} = \dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}\cdot\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} =$

$= \dfrac{2\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} = 2(\sqrt{x}+\sqrt{y}) =$

$= 2\sqrt{x}+2\sqrt{y}.$

- $\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\cdot\dfrac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} =$

$= \dfrac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \dfrac{-h}{hx\sqrt{x+h}+h(x+h)\sqrt{x}}$

$= \dfrac{-1}{x\sqrt{x+h}+(x+h)\sqrt{x}} =$

$= \dfrac{-1}{x\sqrt{x+h}+x^{3/2}+h\sqrt{x}}$