MA109, Activity 22: Rational Functions (Section 4.5, pp. 356-369) **Date:** ____________

`Today's Goal:` We learn how to graph rational functions, that is, functions that are defined as the quotient of two polynomials.

`Assignments:` **Homework (Sec. 4.5):** # 1, 3, 6, 9, 11, 13, 19, 25, 30, 35, 40, 43, 53, 57 (pp. 369-372).

▶ Rational Functions and Asymptotes:

A **rational function** is a function of the form $\boxed{r(x) = \frac{P(x)}{Q(x)}}$ where $P(x)$ and $Q(x)$ are polynomials (that we assume without common factors). When graphing a rational function we must pay special attention to the behavior of the graph near the x-values which make the denominator equal to zero.

Example 1: Find the domain of each of the following rational functions:

$r_1(x) = \frac{x^3+3x^2}{x^2-4}$ $x^2-4\neq0 \Rightarrow (x+2)(x-2)\neq0 \Rightarrow x+2\neq0 \;\&\; x-2\neq0 \Rightarrow x\neq\pm2$

Domain is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

$r_2(x) = \frac{2x+5}{x^2+4}$ $x^2+4>0$ for all x, so domain is $(-\infty,\infty)$

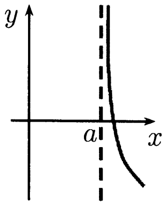
$r_3(x) = \frac{5}{x^2-4x}$ $x^2-4x\neq0 \Rightarrow x(x-4)\neq0 \Rightarrow x\neq0$ and $x\neq4$

Domain is $(-\infty,0)\cup(0,4)\cup(4,\infty)$

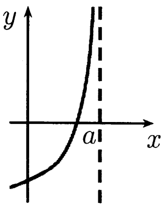
Definition of Vertical and Horizontal Asymptotes:

1. The line $x = a$ is a **vertical asymptote** of $y = r(x)$ if

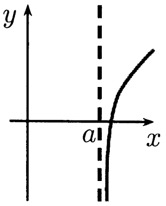
$$y\to\infty \quad \text{or} \quad y\to-\infty \quad \text{as} \quad x\to a^+ \quad \text{or} \quad x\to a^-$$



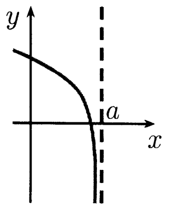
$y\to\infty$
as $x\to a^+$



$y\to\infty$
as $x\to a^-$



$y\to-\infty$
as $x\to a^+$

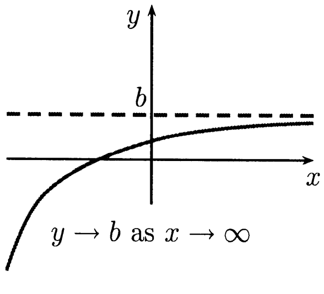


$y\to-\infty$
as $x\to a^-$

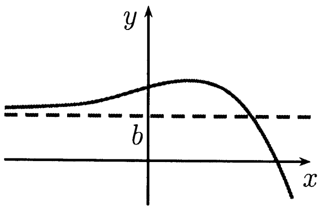
Note: $x\to a^-$ means that "x approaches a from the left";
$x\to a^+$ means that "x approaches a from the right".

2. The line $y = b$ is a **horizontal asymptote** of $y = r(x)$ if

$$y\to b \quad \text{as} \quad x\to\infty \quad \text{or} \quad x\to-\infty$$



$y\to b$ as $x\to\infty$



$y\to b$ as $x\to-\infty$

▶ **Asymptotes of Rational Functions:**

Let $r(x)$ be the rational function:

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}.$$

1. The vertical asymptotes of $r(x)$ are the lines $x = a$, where a is a zero of the denominator.

2. (a) IF $n < m$, then $r(x)$ has horizontal asymptote $y = 0$.

(b) IF $n = m$, then $r(x)$ has horizontal asymptote $y = \frac{a_n}{b_m}$.

(c) IF $n > m$, then $r(x)$ has no horizontal asymptote.

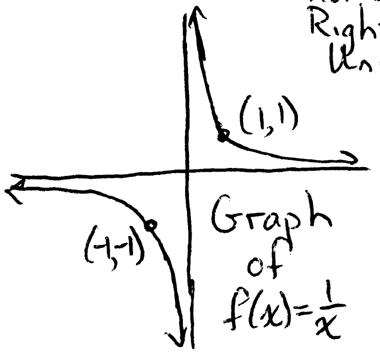
Example 2: Sketch the graphs of:

$$f(x) = \frac{1}{x} \qquad g(x) = \frac{1}{x-2} + 3$$

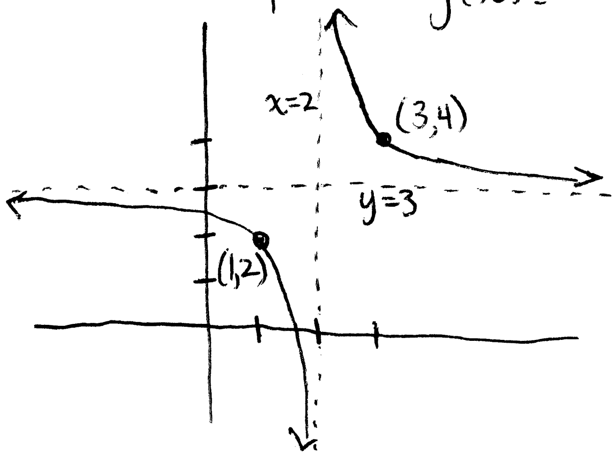
(**Hint:** observe that $g(x) = f(x - 2) + 3$).

Shift Horizontally Right 2 Units.

Shift Vertically Up 3 units.



Graph of $g(x)$:



Example 3: Find the horizontal and vertical asymptotes of each of the following functions:

$r(x) = \frac{6x-2}{x^2+5x-6}$

Vertical:

$x^2+5x-6=0$

$(x+6)(x-1)=0$

$\boxed{x=-6,\ x=1}$

Horizontal: The degree of the numerato is less than the degree of the denominator, So $\boxed{y=0}$ is an asymptote.

$t(x) = \frac{3x^2}{(x+3)(2-x)}$

$t(x)=\frac{3x^2}{-x^2-x+6}$

Vertical:

$x+3=0$ or $2-x=0$

$\boxed{x=-3,\ x=2}$

Horizontal: The degree of the numerator equals the degree of the denominator. $\boxed{y=\frac{3}{-1}=-3}$ is an asymptote

$s(x) = \frac{2x^4+6x^2}{x^2-4}$

Vertical:

$x^2-4=0$

$(x+2)(x-2)=0$

$\boxed{x=-2,\ x=2}$

Horizontal: The degree of the numerator is greater than the degree of the denominator Then $s(x)$ has no horizontal asymptote.

► **Graphing Rational Functions:** 1. **Factor the numerator and denominator.**

2. **Find the x- and the y-intercept(s).**

3. **Find the vertical asymptote(s).**

4. **Find the horizontal asymptote (if any).**

5. **Sketch the Graph.**

Example 4: Graph the rational function $r(x) = \frac{6}{x^2-5x-6}$.

① $r(x) = \frac{6}{(x-6)(x+1)}$

② No x-intercept

y-intercept: $r(0) = \frac{6}{-6} = -1$

③ Vertical Asymptotes:

$x = 6,\ x = -1$

④ Horizontal Asymptote:

$y = 0$

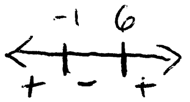
⑤ Test Points:

$r(-2) = \frac{(+)}{(-)(-)} > 0$

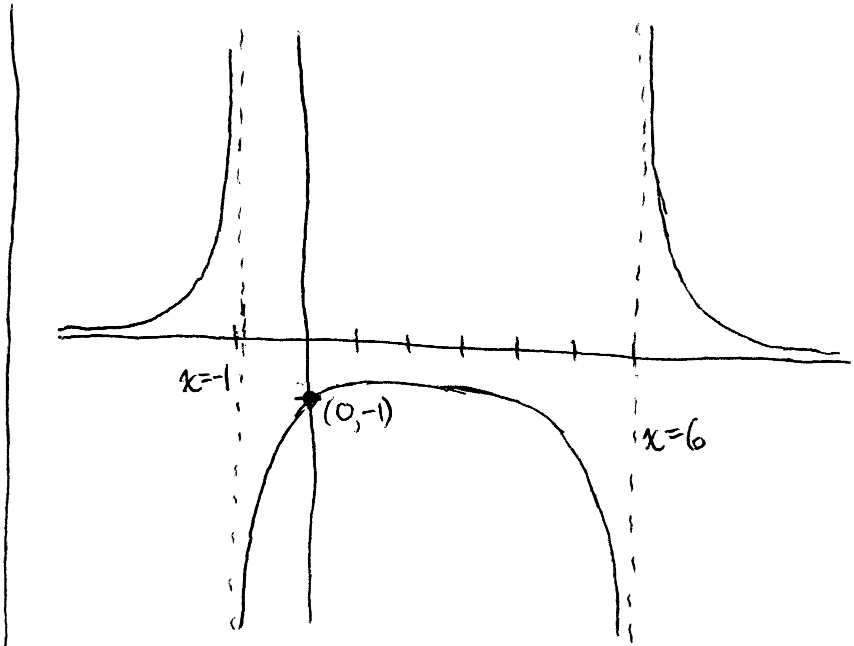
$r(0) = \frac{(+)}{(-)(+)} < 0$

$r(7) = \frac{(+)}{(+)(+)} > 0$

⇒



(Sign of $r(x)$)



Example 5: Graph the rational function $r(x) = \frac{(x-1)(x+2)}{(x+1)(x-3)}$. $= \frac{x^2+x-2}{x^2-2x-3}$

) Done

) $x = 1,\ x = -2$ are x-intercepts

y-intercept: $r(0) = \frac{-2}{-3} = \frac{2}{3}$

) Vertical Asymptotes:

$x = -1,\ x = 3$

) Horizontal Asymptote:

$y = \frac{1}{1} = 1$

) Test points:

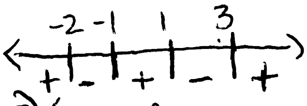
$r(-3) = \frac{(-)(-)}{(-)(-)} > 0$

$r(-1.5) = \frac{(-)(+)}{(-)(-)} < 0$

$r(0) = \frac{(-)(+)}{(+)(-)} > 0$

$r(2) = \frac{(+)(+)}{(+)(-)} < 0$

$r(4) = \frac{(+)(+)}{(+)(+)} > 0$



⇒ (Sign of $r(x)$)