MA109, Activity 23: Exponential Functions (Section 5.1, pp. 384-392) Date: ____________

Today's Goal: We introduce a new class of functions called *exponential functions*. Exponential functions are appropriate for modeling such natural processes as population growth for all living things and radioactive decay. We also use exponential functions to find compound interest. Even the Gateway Arch in St. Louis is shaped (for optimal distribution of internal structural forces) in the form of the graph of a combination of exponential functions!

Assignments: **Homework** (Sec. 5.1): # 2, 3, 7, 10, 11, 14, 17, 19, 21, 25, 32, 59, 67 (pp. 392-396).

▸ **Exponential Functions:**

Let $a > 0$ be a positive number with $a \neq 1$. The **exponential function with base** a is defined by

$$f(x) = a^x$$

for all real numbers x.

In Section P.4, a^x is defined for $a > 0$ and x a rational number. So, what does, for instance, $5^{\sqrt{2}}$ mean? When x is irrational, we successively approximate x by rational numbers. For instance, as

$$\sqrt{2} \approx 1.41421\ldots$$

we successively approximate $5^{\sqrt{2}}$ with

$$5^{1.4},\quad 5^{1.41},\quad 5^{1.414},\quad 5^{1.4142},\quad 5^{1.41421},\ldots$$

In practice, we simply use our calculator and find out

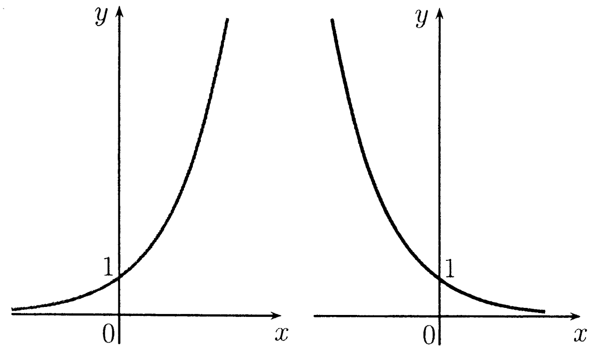
$$5^{\sqrt{2}} \approx 9.73851\ldots$$

▸ **Graphs of Exponential Functions:**

The **exponential function**

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$ $\qquad$ $f(x) = a^x$ for $0 < a < 1$

Example 1: Let $f(x) = 2^x$. Evaluate the following: Calculator $2 \wedge (-1/3)$

$f(2) = 2^2 = 4$

$f(-1/3) = 2^{-1/3} = \frac{1}{2^{1/3}} = \frac{1}{\sqrt[3]{2}} \approx .793700526\ldots$

$f(\pi) = 2^{\pi} \approx 8.824977827\ldots$

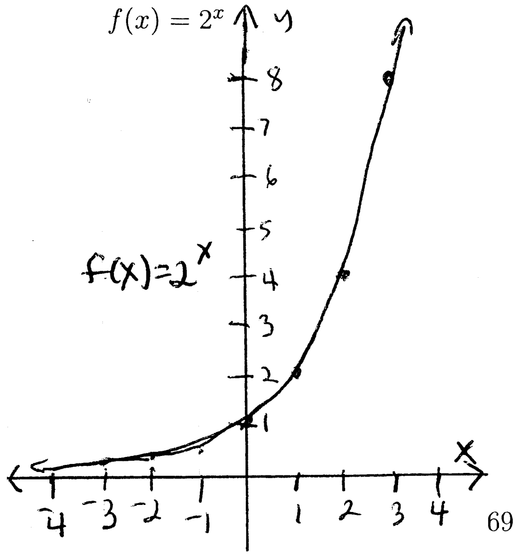
$f(-\sqrt{3}) = 2^{-\sqrt{3}} = \frac{1}{2^{\sqrt{3}}} \approx .3010237439\ldots$

calculator $2 \wedge (-\sqrt{3})$

Example 2: Draw the graph of each function:

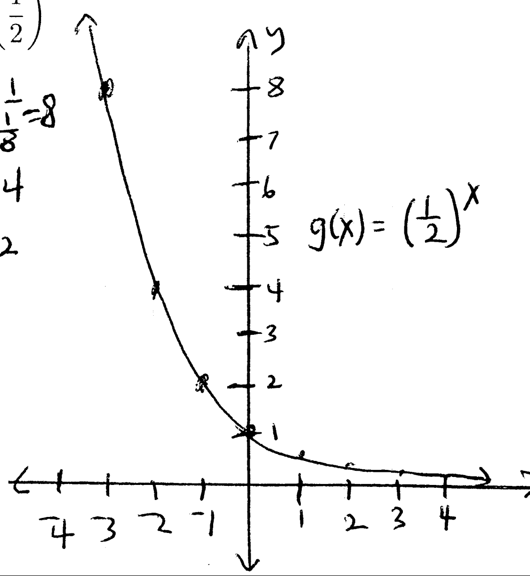
$f(x) = 2^x$

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

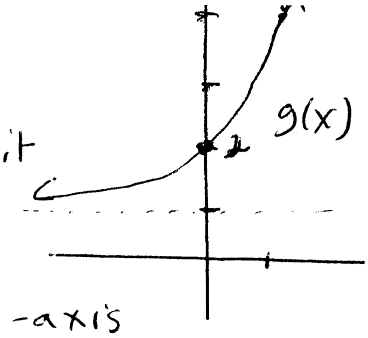


$g(x) = \left(\frac{1}{2}\right)^x$

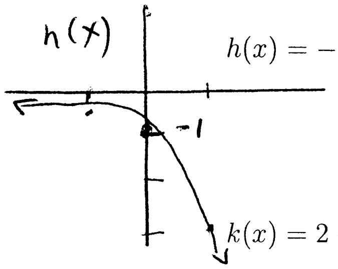
x	$y = \left(\frac{1}{2}\right)^x$
-3	$\left(\frac{1}{2}\right)^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}} = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = \frac{1}{\left(\frac{1}{2}\right)} = 2$
0	$\left(\frac{1}{2}\right)^0 = 1$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$



Example 3: Use the graph of $f(x) = 3^x$ to sketch the graph of each function:

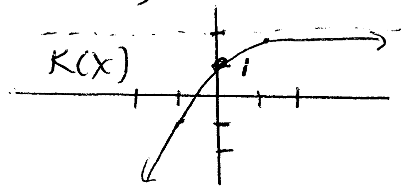


$g(x) = 1 + 3^x$ Shift the graph of $f(x) = 3^x$ up 1 unit



$h(x) = -3^x$ Reflect the graph of $f(x) = 3^x$ in the x-axis

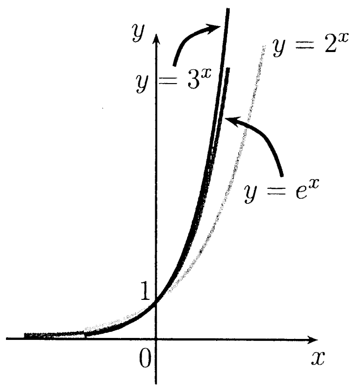
$k(x) = 2 - 3^{-x}$ Reflect the graph of $f(x) = 3^x$ in the x-axis, in the y-axis and then shift this up 2 units



The most important base is the number denoted by the letter e. The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes very large. Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

> ► **The Natural Exponential Function:**
>
> The **natural exponential function** is the exponential function
>
> $$f(x) = e^x$$
>
> with base e. It is often referred to as *the* exponential function.

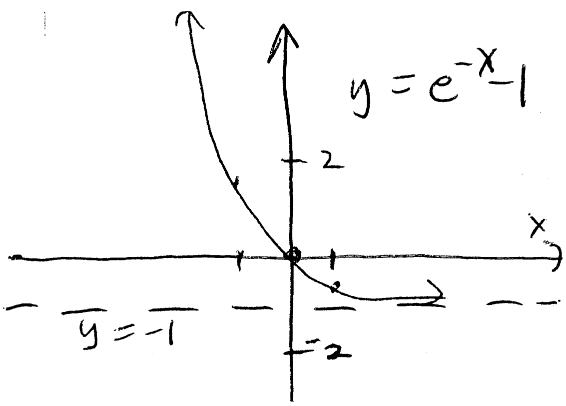
Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

Example 4: Sketch the graph of $y = e^{-x} - 1$.

Reflect $y = e^x$ in the y-axis
Shift this down 1 unit



Example 5:

When a certain drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50\,e^{-0.2t}.$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

$D(3) = 50\,e^{-0.2(3)} \approx 27.4405818\ldots$

► **Compound Interest:**

Compound interest is calculated by the formula:

$$\boxed{A(t) = P\left(1+\frac{r}{n}\right)^{nt}}$$

where

$$\begin{array}{ccl} A(t) & = & \text{amount after } t \text{ years} \\ P & = & \text{principal} \\ r & = & \text{interest rate per year} \\ n & = & \text{number of times interest is compounded per year} \\ t & = & \text{number of years} \end{array}$$

Continuously Compounded interest is calculated by the formula:

$$\boxed{A(t) = P\,e^{rt}}$$

where

$$\begin{array}{ccl} A(t) & = & \text{amount after } t \text{ years} \\ P & = & \text{principal} \\ r & = & \text{interest rate per year} \\ t & = & \text{number of years} \end{array}$$

Proof: The interest paid increases as the number n of compounding periods increases. If $m = \dfrac{n}{r}$, then:

$$P\left(1+\frac{r}{n}\right)^{nt} = P\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

But as m becomes large, the quantity $(1+1/m)^m$ approaches the number e. Thus, we obtain the formula for the continuously compounded interest.

Example 6:

Suppose you invest \$2,000 at an annual rate of 12% ($r = 0.12$) compounded quarterly ($n = 4$). How much money would you have one year later? What if the investment was compounded monthly ($n = 12$)?

Compounded Quarterly $A(1) = 2000\left(1+\frac{.12}{4}\right)^{4(1)} = 2000\,(1.03)^4 \approx 2251.02$

Compounded Monthly $A(1) = 2000\left(1+\frac{.12}{12}\right)^{12(1)} = 2000\,(1.01)^{12} \approx 2253.65$

Example 7:

Suppose you invest \$2,000 at an annual rate of 9% ($r = 0.09$) compounded continuously. How much money would you have after three years?

$A(3) = 2000\, e^{.09(3)} \approx 2619.93$