MA109, Activity 24: Logarithmic Functions (Section 5.2, pp. 398-405) Date: ____________

Today's Goal: We study the inverse of exponential functions, that is, *logarithmic functions*.

Assignments: **Homework (Sec. 5.2):** # 1,3,6,7,10,12,14,17,19,23,29,35,37,41,44,59,61 (pp. 406-408).

▶ **Logarithmic Functions:** Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test. Thus, it has an inverse function (see Activity 17). The inverse function $f^{-1}(x)$ is called the *logarithmic function with base a* and is denoted by $\log_a x$.

Definition: Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a x \iff a^y = x.$$

In other words, $\log_a x$ is the exponent to which the base a must be raised to give x.

Properties of Logarithms:

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

Example 1: Change each exponential expression into an equivalent expression in logarithmic form:

$5^3 = b$

$\log_5 b = 3$

$a^6 = 15$

$\log_a 15 = 6$

$e^{t+1} = 0.5$

$\log_e 0.5 = t + 1$

Example 2: Change each logarithmic expression into an equivalent expression in exponential form:

$\log_3 81 = 4$

$3^4 = 81$

$\log_8 4 = \frac{2}{3}$

$8^{2/3} = 4$

$\log_e(x - 3) = 2$

$e^2 = x - 3$

Example 3: Evaluate each of the following expressions:

$\log_4 64$

$\log_4 64 = \log_4 4^3 = 3$

* property 3

$\log_{10} \sqrt{10}$

$\log_{10} \sqrt{10} = \log_{10} 10^{1/2}$

$= 1/2$

by property 3

$\log_2 \frac{1}{32}$

$\log_2 \frac{1}{32} = \log_2 \frac{1}{2^5} = \log_2 2^{-5}$

$= -5$

by property 3

$3^{\log_3 7}$

$3^{\log_3 7} = 7$

by property 4

$\log_8 64$

$\log_8 64 = \log_8 8^2$

$= 2$

by property 3

$\log_3 1$

$\log_3 1 = 0$

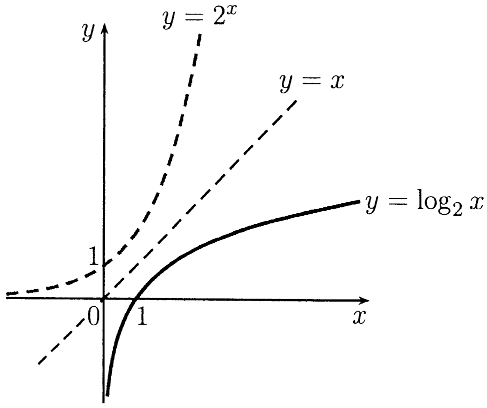
by property 1

Remark:

If a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. THUS, the function $y = \log_a x$ is defined for $x > 0$ and has range equal to $\mathbb{R}$. More precisely:

▶ **Graphs of Logarithmic Functions:**

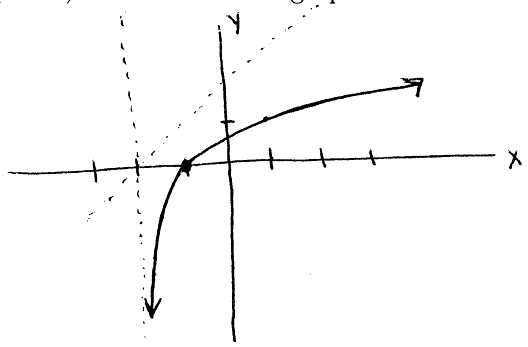
The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. (The picture below shows a typical case with $a > 1$.)



The point $(1, 0)$ is on the graph of $y = \log_a x$ (as $\log_a 1 = 0$) and the y-axis is a vertical asymptote.

Example 4: Find the domain of the function $f(x) = \log_3(x+2)$ and sketch its graph.

since $y = \log_a x$ is defined for $x > 0$
then $f(x)$ is defined for $x+2 > 0$
or, $x > -2$
hence, the domain of $f(x)$ is
$(-2, \infty)$



▶ **Common Logarithms:**

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\boxed{\log x := \log_{10} x.}$$

Example 5 (Bacteria Colony):

A certain strain of bacteria divides every three hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\,\frac{\log(N/50)}{\log 2}.$$

Find the time required for the colony to grow to a million bacteria.

ie: find t when $N = 1{,}000{,}000$

$$t = 3\,\frac{\log\left(\frac{1{,}000{,}000}{50}\right)}{\log 2} = 3\,\frac{\log(20000)}{\log(2)} \approx 42.9 \text{ hrs.}$$

► **Natural Logarithms:** Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e (see Activity 23).

Definition: The logarithm with base e is called the **natural logarithm** and is denoted by ln:

$$\boxed{\ln x := \log_e x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \quad \Longleftrightarrow \quad e^y = x.$$

Properties of Natural Logarithms:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Example 6: Evaluate each of the following expressions:

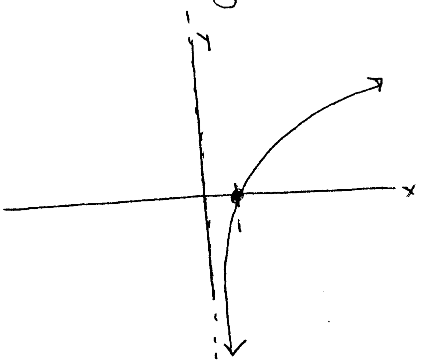
$\ln e^9 = 9$ by property 3

$\ln \frac{1}{e^4} = \ln e^{-4} = -4$ by property 3

$e^{\ln 2} = 2$ by property 4

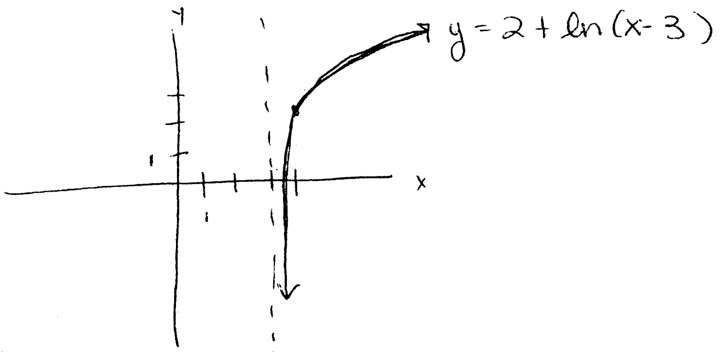
Example 7: Graph the function $y = 2 + \ln(x - 3)$.

Consider $g(x) = \ln(x)$



Note $f(x) = 2 + \ln(x-3) = 2 + g(x-3)$

So, the graph of $f(x)$ is the graph of $g(x)$ shifted right 3 and up 2.



Example 8: Find the domain of the function $f(x) = 2 + \ln(10 + 3x - x^2)$.

Recall the domain of $g(x) = \ln(x)$ is $(0, \infty)$ thus $f(x)$ is only defined for values of x such that

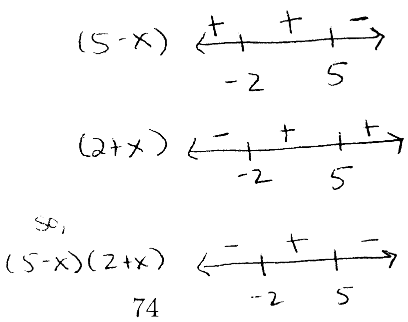
$$10 + 3x - x^2 > 0$$

Solving this inequality:

$(5 - x)(2 + x) > 0$

$5 - x = 0$ $\quad$ $2 + x = 0$

$x = 5$ $\quad$ $x = -2$



hence, the solution to the inequality is $(-2, 5)$

So, the domain of $f(x)$ is $(-2, 5)$