MA109, Activity 25: Laws of Logarithms (Section 5.3, pp. 409-413) **Date:** ______

`Today's Goal:`	`Logarithmic functions are used to model a variety of situations involving human behavior. We study properties of logarithms: These properties give logarithmic functions a wide range of applications (see Activity 40).`
`Assignments:`	**Homework (Sec. 5.3):** `# 1, 5, 8, 12, 15, 19, 20, 23, 24, 31, 40, 44, 45, 49, 55 (pp. 413-415).`

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

▶ **Laws of Logarithms:**

1. $\log_a(AB) = \log_a A + \log_a B$;

2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$;

3. $\log_a(A^C) = C \log_a A$.

Proof of Law 1: Let us set

$$\log_a A = u \quad \text{and} \quad \log_a B = v.$$

When written in exponential form, they become

$$a^u = A \quad \text{and} \quad a^v = B.$$

Thus:
$$\begin{aligned} \underline{\log_a(AB)} &= \log_a(a^u\, a^v) \\ &= \log_a(a^{u+v}) \\ &\overset{\text{why?}}{=} u+v \\ &= \underline{\log_a A + \log_a B}. \end{aligned}$$

In a similar fashion, one can prove **2.** and **3.**

Example 1: Evaluate each expression:

$\log_5 5^9$

$= 9\log_5 5 = 9$

$\log_3 7 + \log_3 2$

$= \log_3 7\cdot 2 = \log_3 14$

$\log_3 16 - 2\log_3 2$

$= \log_3 16 - \log_3 2^2$

$= \log_3 16 - \log_3 4$

$= \log_3 16/4 = \log_3 4$

$\ln\left(\ln e^{(e^{200})}\right)$

$= \ln(e^{200}\ln e)$

$= \ln e^{200} = 200\ln e = 200$

$\log_3 100 - \log_3 18 - \log_3 50$

$= \log_3 \frac{100}{18} - \log_3 50 = \log_3 \left(\frac{100}{18}\right)/50$

$= \log_3 \frac{100}{900} = \log_3 \frac{1}{9} = \log_3 3^{-2} = -2\log_3 3 = -2$

▶ **Expanding and Combining Logarithmic Expressions:**

Example 2: Use the Laws of Logarithms to expand each expression:

$\log_2(2x)$

$= \log_2 2 + \log_2 x$

$= 1 + \log_2 x$

$\log_5(x^2(4-5x))$

$= \log_5 x^2 + \log_5(4-5x)$

$= 2\log_5 x + \log_5(4-5x)$

$\log\left(x\sqrt{\frac{y}{z}}\right)$

$= \log x + \log\sqrt{\frac{y}{z}}$

$= \log x + \log\left(\frac{y}{z}\right)^{1/2}$

$= \log x + \frac{1}{2}\log\frac{y}{z}$

$= \log x + \frac{1}{2}(\log y - \log z)$

Example 3: Use the Laws of Logarithms to combine the expression

$$\log_a b + c\log_a d - r\log_a s$$

into a single logarithm.

$$\log_a b + c\log_a d - r\log_a s = \log_a b + \log_a d^c - \log_a s^r$$

$$= \log_a bd^c - \log_a s^r$$

$$= \log_a \frac{bd^c}{s^r}$$

Example 4: Use the Laws of Logarithms to combine the expression

$$\ln 5 + \ln(x+1) + \frac{1}{2}\ln(2-5x) - 3\ln(x-4) - \ln x$$

into a single logarithm.

$$\ln 5 + \ln(x+1) + \frac{1}{2}\ln(2-5x) - 3\ln(x-4) - \ln x = \ln 5 + \ln(x+1) + \ln(2-5x)^{1/2} - \ln(x-4)^3 - \ln x$$

$$= \ln\left(5(x+1)(2-5x)^{1/2}\right) - \ln(x-4)^3 - \ln x$$

$$= \ln\left(\frac{5(x+1)(2-5x)^{1/2}}{(x-4)^3 x}\right)$$

Example 5 (Forgetting):

Ebbinghaus's Law of Forgetting states that if a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1),$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve the equation for P.

(b) Use Ebbinghaus's Law of Forgetting to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume $c = 0.3$.

(a) $\log P = \log P_0 - c\log(t+1) \iff P = 10^{(\log P_0 - c\log(t+1))}$

$$\Rightarrow P = 10^{\log \frac{P_0}{(t+1)^c}}$$

$$\Rightarrow P = \frac{P_0}{(t+1)^c}$$

(b) $P = \dfrac{80}{(24+1)^{0.3}} \approx 30.458$

Example 6 (Biodiversity):

Some biologists model the number of species S in a fixed area A (such as an island) by the **Species-Area relationship**

$$\log S = \log c + k \log A,$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part (a) to show that if $k = 3$ then doubling the area increases the number of species eightfold.

(a) $\log S = \log c + k \log A \iff S = 10^{(\log c + k\log A)}$

$\Rightarrow S = 10^{\log cA^k}$

$\Rightarrow S = cA^k$

(b) $k=3$ & $A \to 2A$

$S = c(2A)^3 = c2^3A^3 = \underline{8}cA^3$

For $A \to A$ & $k=3$ $\quad S = cA^3$

▶ **Change of Base:**

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

Example 7:

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct up to five decimal places:

$\log_5 2 = \dfrac{\ln 2}{\ln 5} \approx 0.43068$

$\log_4 125 = \dfrac{\ln 125}{\ln 4} \approx 3.48289$

$\log_{\sqrt{3}} 5 = \dfrac{\ln 5}{\ln \sqrt{3}} \approx 2.92994$