MA109, Activity 26: Exponential and Logarithmic Equations (Section 5.4, pp. 415-424) **Date:** ________

Today's Goal: We learn how to solve equations that involve exponential and logarithmic functions. In Activity 27, we will use the techniques that we develop here to solve applied problems.

Assignments: **Homework (Sec. 5.4):** # 1, 5, 11, 16, 19, 23, 27, 31, 37, 40, 43, 47, 49, 67, 71, 77 (pp. 424-426).

▶ **Exponential Equations:**

An exponential equation is one in which the variable occurs in the exponent. For example,

$$3^{x-2} = 7.$$

We take the (either common or natural) logarithm of each side and then use the Laws of Logarithms to 'bring down the variable' from the exponent:

$$\log(3^{x+2}) = \log 7 \quad \rightsquigarrow \quad (x+2)\log 3 = \log 7 \quad \rightsquigarrow$$

$$x + 2 = \frac{\log 7}{\log 3} \quad \rightsquigarrow \quad x = \frac{\log 7}{\log 3} - 2 \approx -0.228756.$$

Guidelines for Solving Exponential Equations:

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to 'bring down the exponent.'
3. Solve for the variable.
4. Check your answer.

Example 1: Solve the equations:

$3 \cdot 4^x = 18$

$4^x = 6$

$\ln(4^x) = \ln(6)$

$x \cdot \ln(4) = \ln(6)$

$x = \frac{\ln(6)}{\ln(4)} \approx 1.292$

$3^{x+4} = 2^{1-2x}$

$\ln(3^{x+4}) = \ln(2^{1-2x})$

$(x+4)\cdot\ln(3) = (1-2x)\cdot\ln(2)$

$x\cdot\ln(3) + 4\cdot\ln(3) = \ln(2) - 2x\cdot\ln(2)$

$x\cdot\ln(3) + 2x\cdot\ln(2) = \ln(2) - 4\ln(3)$

$x\cdot(\ln(3) + 2\cdot\ln(2)) = \ln(2) - 4\ln(3)$

$x = \frac{\ln(2) - 4\ln(3)}{\ln(3) + 2\ln(2)} \approx -1.921$

Example 2: Solve the following exponential equations of quadratic type:

$9^x - 3^x = 72$

$(3^2)^x - 3^x = 72$

$3^{2x} - 3^x = 72$

Let $u = 3^x$

$u^2 - u = 72$

$u^2 - u - 72 = 0$

$(u+9)(u-8) = 0$

So $3^x = 9$ or $3^x = -8$ can't happen

Thus $3^x = 9$ and $x = 2$

$4^x - 3(4^{-x}) = 2$

multiply both sides by 4^x

$(4^x)^2 - 3 = 2(4^x)$

Let $u = 4^x$

$u^2 - 3 = 2u$

$u^2 - 2u - 3 = 0$

$(u-3)(u+1) = 0$

So $u = 3$ or $u = -1$

$4^x = 3$ or $4^x = -1$ ← can't happen

Thus $4^x = 3$ and $x = \frac{\ln(3)}{\ln(4)}$

Example 3: Solve the equation $x^2e^x + xe^x - 6e^x = 0$.

Factor out e^x to get $e^x(x^2+x-6)=0$

$e^x(x+3)(x-2)=0$

Since e^x is never 0, we know that $x=-3$ or $x=2$

► **Logarithmic Equations:**

A logarithmic equation is one in which a logarithm of the variable occurs. For example,

$$\log_2(25-x) = 3.$$

To solve for x, we write the equation in exponential form, and then solve for the variable:

$$25 - x = 2^3 \quad \rightsquigarrow \quad 25 - x = 8 \quad \rightsquigarrow \quad x = 17.$$

Alternatively, we raise the base, 2, to each side of the equation; we then use the Laws of Logarithms:

$$2^{\log_2(25-x)} = 2^3 \quad \rightsquigarrow \quad 25 - x = 2^3 \quad \rightsquigarrow \quad x = 17.$$

Guidelines for Solving Logarithmic Equations:

1. Isolate the logarithmic term on one side of the equation; you may first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable. Check your answers!

Example 4: Solve the following equations:

$$2\log_7 x = \log_7 16$$

$\log_7(x^2)=\log_7(16)$

$7^{(\log_7(x^2))} = 7^{(\log_7(16))}$

$x^2=16$

$x=4$ or $x=-4$

But since we can't take $\log_7(-4)$, $\boxed{x=4}$ is the only answer

$$\log_2(x+3) = \log_2(x-3) + \log_3 9 + 4^{\log_4 3}$$

First, $\log_3 9 = 2$ and $4^{\log_4 3}=3$

So $\log_2(x+3)-\log_2(x-3) = 2+3$

$\log_2\left(\frac{x+3}{x-3}\right)=5$

$\frac{x+3}{x-3}=2^5$

$x+3=32(x-3)$

$x+3=32x-96$

$99=31x$, so $x=\frac{99}{31}$

Example 5: Solve the following equations:

$$\log_6(x+5) + \log_6 x = 2$$

$\log_6[(x+5)x]=2$

$\log_6(x^2+5x)=2$

$x^2+5x=6^2$

$x^2+5x=36$

$x^2+5x-36=0$

$(x+9)(x-4)=0$

So $x=-9$ or $x=4$

But since $\log_6(-9)$ is undefined, $\boxed{x=4}$

$$\log(x^3) = (\log x)^3$$

$3\log(x)=(\log(x))^3$

Let $u=\log(x)$

$3u=u^3$

$0=u^3-3u$

$0=u(u^2-3)$

So $u=0$ or $u^2-3=0$

$\log x=0$ $\quad$ $u=\pm\sqrt{3}$

$\log x=\pm\sqrt{3}$

So $x=1$ or $x=10^{\pm\sqrt{3}}$

Example 6 (Genetic Mutation): The basic source of genetic diversity is mutation (that is, changes in the chemical structure of genes). If genes mutate at a constant rate m (with $0 < m < 1$) and if other evolutionary forces are neglegible, then the frequency F of the original gene after t generations is

$$F = F_0(1-m)^t,$$

where F_0 is the frequency at $t = 0$.

(a) Solve the above equations for t, using log.

(b) If $m = 5 \times 10^{-5}$, after how many generations is $F/F_0 = 1/2$?

a) $\frac{F}{F_0} = (1-m)^t$

$\log\left(\frac{F}{F_0}\right) = \log\left[(1-m)^t\right]$

$\log\left(\frac{F}{F_0}\right) = t \cdot \log(1-m)$

$t = \frac{\log\left(\frac{F}{F_0}\right)}{\log(1-m)}$

b) $\frac{F}{F_0} = \frac{1}{2}$, $m = .00005$

So $t = \frac{\log\left(\frac{1}{2}\right)}{\log(1-.00005)}$

$t = \frac{\log\left(\frac{1}{2}\right)}{\log(0.99995)}$

$t \approx \frac{-.30103}{-2.1715 \times 10^{-5}}$

$t \approx 13,862.597$

▶ **Compound Interest:** We recall the formulæ we found in Activity 36. If a principal P is invested at an interest rate r for a period of t years, then the amount $A(t)$ of the investment is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{(if compounded } n \text{ times per year)}$$

$$A(t) = P\,e^{rt} \quad \text{(if compounded continuously).}$$

We can use logarithms to determine the time it takes for the principal to increase to a given amount.

Example 7: How long will it take for an investment of \$10,000 to double in value if the interest rate is 8.5% per year, compounded continuously?

$P = 10,000$

$r = .085$

$A(t) = 20,000$

we must find t

$20,000 = 10,000\,e^{.085 t}$

$2 = e^{.085 t}$

$\ln(2) = .085t$

$t = \frac{\ln(2)}{.085} \approx 8.15467$

Example 8: Suppose that \$1,000 is invested in an account where the interest is compounded semi-annually. If the account is worth \$1,435.77 in four years, then what is the interest rate?

$P = 1000$

$n = 2$

$A(t) = 1,435.77$

$t = 4$

must find r

$1435.77 = 1000\left(1 + \frac{r}{2}\right)^{2 \cdot 4}$

$1.43577 = \left(1 + \frac{r}{2}\right)^8$

$(1.43577)^{1/8} = \left[\left(1 + \frac{r}{2}\right)^8\right]^{1/8}$

$1.04625 = 1 + \frac{r}{2}$

$.04625 = \frac{r}{2}$

$r = 0.0925$ or 9.25%