Today's Goal: We learn how to solve quadratic (or second-degree) equations.

Assignments: **Homework (Sec. 1.3):** # 1, 8, 13, 19, 23, 29, 38, 43, 71, 75, 79, 89, 91 (pp. 105-108).

A **quadratic equation** is an equation of the form:

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

Zero-Product Property: For any $A, B \in \mathbb{R}$: $AB = 0$ if and only if $A = 0$ or $B = 0$.

Example 1: Solve the following equations by factoring:

- $x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$x - 3 = 0$, $\boxed{x = 3}$

or

$x - 4 = 0$, $\boxed{x = 4}$

- $2z^2 = z + 3$

$2z^2 - z - 3 = 0$

$(2z - 3)(z + 1) = 0$

$2z - 3 = 0$, $\boxed{z = 3/2}$

or

$z + 1 = 0$, $\boxed{z = -1}$

▶ **Solving Quadratic Equations by Completing the Square:**

If a quadratic equation is of the form

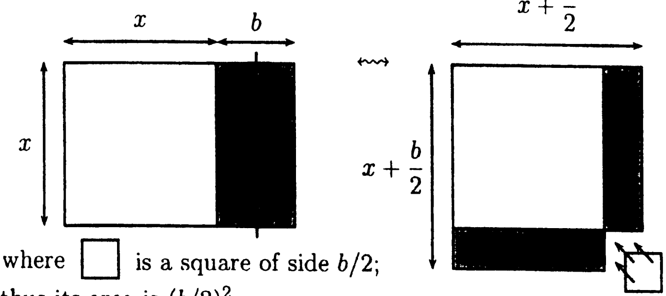
$$(x \pm \alpha)^2 = \beta,$$

we can solve it by taking the square root of each side. So, if a quadratic equation does not factor readily ... we solve it by completing the square!

Completing the Square: To make a perfect square out of $x^2 + bx$, add the square of half the coefficient of x, that is $(b/2)^2$. Thus:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Geometric Interpretation of Completing the Square: This interpretation goes back to the Babylonian scribes, who fully used the "cut-and-paste" geometry developed by the ancient surveyors (ca. 1700 BC). Here, x and b are positive as they represent lengths:



where $\square$ is a square of side $b/2$; thus its area is $(b/2)^2$.

Example 2: Solve each equation by completing the square:

- $x^2 + 4x - 6 = 0$

$x^2 + 4x = 6$

$x^2 + 4x + 4 = 6 + 4$

$(x+2)^2 = 10$

$x + 2 = \pm\sqrt{10}$

$x = -2 \pm \sqrt{10}$

- $3x^2 - 6x - 1 = 0$

$x^2 - 2x - \frac{1}{3} = 0$

$x^2 - 2x = 1/3$

$x^2 - 2x + 1 = 1/3 + 1$

$(x-1)^2 = 4/3$

$x - 1 = \pm\sqrt{4/3}$

$x = 1 \pm \sqrt{4/3}$

[redacted]

We can use the technique of completing the square to derive a formula for the general quadratic equation:

$$ax^2 + bx + c = 0$$

(where $a \neq 0$). We obtain the following:

> ▶ **The Quadratic Formula:**
>
> The roots x_1 and x_2 of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are:
>
> $$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof:

$$ax^2 + bx + c = 0$$
$$\Updownarrow$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$\Updownarrow$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\Updownarrow$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$
$$\Updownarrow$$
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$
$$\Updownarrow$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 3: Find all solutions of each equation:

- $3x^2 + 7x + 4 = 0$

$$x_1, x_2 = \frac{-7 \pm \sqrt{7^2 - 4(3)(4)}}{2(3)}$$

$$x_1, x_2 = \frac{-7 \pm \sqrt{1}}{6}$$

$$\boxed{x_1 = \frac{-7+1}{6} = -1}, \quad \boxed{x_2 = \frac{-7-1}{6} = \frac{-8}{6}}$$

- $2t^2 + 5t + 3 = 0$

$$x_1, x_2 = \frac{-5 \pm \sqrt{5^2 - 4(2)(3)}}{2(2)}$$

$$x_1, x_2 = \frac{-5 \pm \sqrt{1}}{4}$$

$$\boxed{x_1 = \frac{-5+1}{4} = -1}, \quad \boxed{x_2 = \frac{-5-1}{4} = \frac{-6}{4}}$$

- $x = 1 - \frac{4}{x}$

$$x^2 = x - 4$$

$$x^2 - x + 4 = 0$$

$$x_1, x_2 = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(4)}}{2(1)}$$

$$x_1, x_2 = \frac{1 \pm \sqrt{-15}}{2} \Longrightarrow \boxed{\text{NO REAL ROOTS}}$$

- $9 + \frac{3}{x} - \frac{2}{x^2} = 0$

$$9x^2 + 3x - 2 = 0$$

$$x_1, x_2 = \frac{-3 \pm \sqrt{9 - 4(9)(-2)}}{2(9)}$$

$$x_1, x_2 = \frac{-3 \pm \sqrt{81}}{18}$$

$$x_1, x_2 = (-3 \pm 9)/18$$

$$\boxed{x = \frac{6}{18} \text{ or } x = -1}$$

> ▶ **The Discriminant:**
>
> The **discriminant** D of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, is:
>
> $$\boxed{D = b^2 - 4ac}$$
>
> **1.** If $D > 0$ the eq. has 2 distinct real roots.
>
> **2.** If $D = 0$ the eq. has exactly 1 real root.
>
> **3.** If $D < 0$ the eq. has no real roots.

Example 4: Use the discriminant to determine how many real roots each equation has. Do not solve the equation.

- $3x^2 - 5x + 1 = 0$

$$D = (-5)^2 - 4(3)(1)$$

$$D = 13$$

$$13 > 0 \Rightarrow \boxed{2 \text{ real roots}}$$

- $x^2 = 6x - 10$

$$x^2 - 6x + 10 = 0$$

$$D = (-6)^2 - 4(1)(10)$$

$$D = 36 - 40$$

$$D = -4$$

$$-4 < 0$$

$$\boxed{\text{NO REAL ROOTS}}$$

[redacted]

Example 5:

Find all nonzero values of k that ensure that the equation

$$kx^2 + 36x + k = 0$$

has exactly one root (solution).

$D = 0 \Rightarrow$ 1 REAL SOLUTION

$0 = 36^2 - 4(k)(k)$

$0 = 36^2 - 4k^2$

$4k^2 = 36^2$

$k^2 = 324$

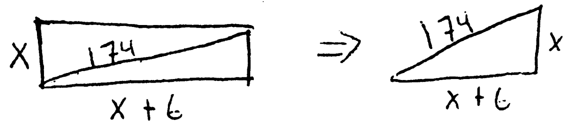
$k = \pm\sqrt{324}$

$\boxed{k = \pm 18}$

▶ **Modeling with Quadratic Equations:**

The principles discussed in Activity 2 for setting up equations as models are useful here as well.

Example 6 (Dimension of a Lot): A parcel of land is 6 ft longer than it is wide. Each diagonal from one corner to the opposite one is 174 ft long. What are the dimensions of the parcel?



$a^2 + b^2 = c^2$

$(x+6)^2 + x^2 = 174^2$

$x^2 + 12x + 36 + x^2 = 174^2$

$2x^2 + 12x - 30240 = 0$

$x_1, x_2 = \dfrac{-12 \pm \sqrt{242064}}{4}$

$x_1, x_2 = \dfrac{-12 \pm 492}{4}$

$x_1 = \dfrac{-12 + 492}{4} = 120$

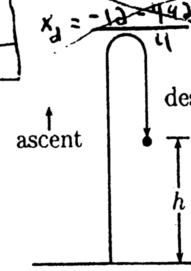
$\cancel{x_2 = \dfrac{-12 - 492}{4} = -126}$ (Dimension MUST BE Positive)

$\boxed{\text{Dimensions Are } 120 \times 126}$

Example 7 (Falling-Body Problem):

An object is thrown straight upward at an initial speed of 400 ft/s. From Physics, it is known that, after t seconds, it reaches a height of h feet given by the formula:

$$h = -16t^2 + 400t.$$



(*a*) When does the object fall back to ground level?

$h = 0$

$0 = -16t^2 + 400t$

$x_1, x_2 = \dfrac{-400 \pm \sqrt{400^2 - 4(-16)(0)}}{2(-16)}$

$x_1, x_2 = \dfrac{-400 \pm \sqrt{400^2}}{-32}$

$x_1, x_2 = \dfrac{-400 \pm 400}{-32}$

$x_1 = \dfrac{-400 + 400}{-32} = 0$

$\boxed{x_2 = \dfrac{-400 - 400}{-32} = 25}$

(*b*) When does it reach a height of 1,600 ft?

$1600 = -16t^2 + 400t$

$0 = -16t^2 + 400t - 1600$

$x_1 x_2 = \dfrac{-400 \pm \sqrt{400^2 - 4(-16)(-1600)}}{2(-16)}$

$x_1 x_2 = \dfrac{-400 \pm \sqrt{57600}}{-32}$

$x_1 x_2 = \dfrac{-400 \pm 240}{-32}$

$\boxed{x_1 = \dfrac{-400 + 240}{-32} = 5}$

$\boxed{x_2 = \dfrac{-400 - 240}{-32} = 20}$

(*c*) When does it reach a height of 1 mi? (1 mi = 5,280 ft)

$5280 = -16t^2 + 400t$

$0 = -16t^2 + 400t - 5280$

$x_1, x_2 = \dfrac{-400 \pm \sqrt{400^2 - 4(-16)(-5280)}}{2(-16)}$

$x_1, x_2 = \dfrac{-400 \pm \sqrt{-177920}}{-32}$

$\Rightarrow$ $\boxed{\text{NO REAL ROOTS}}$

(*d*) How high is the highest point the object reaches?

OBJECT reaches the highest point once.

So Discriminant $= 0$

$0 = -16t^2 + 400t - h$

$D = 400^2 - 4(-16)(-h) = 0$

$400^2 = 64h$

$\boxed{2{,}500 = h}$