10 Exponential and Logarithmic Functions

Concepts:

- Rules of Exponents
- Exponential Functions
  - Power Functions vs. Exponential Functions
  - The Definition of an Exponential Function
  - Graphing Exponential Functions
  - Exponential Growth and Exponential Decay
- Compound Interest
- Logarithms
  - Logarithms with Base a
    * Definition
    * Exponential Notation vs. Logarithmic Notation
    * Evaluating Logarithms
    * Graphs of Logarithms
    * Domain of Logarithms
  - Properties of Logarithms
    * Simplifying Logarithmic Expressions
    * Using the Change of Base Formula to Find Approximate Values of Logarithms
- Solving Exponential and Logarithmic Equations

(Chapter 5)

10.1 Rules of Exponents

The following are to remind you of the rules of exponents. You are expected to know how to use them. To review, see section 5.1 in your textbook.

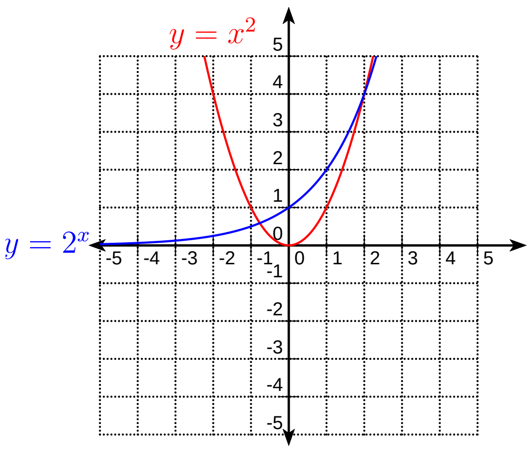
Let c be a nonnegative real number, and let r and s be any rational numbers. Then

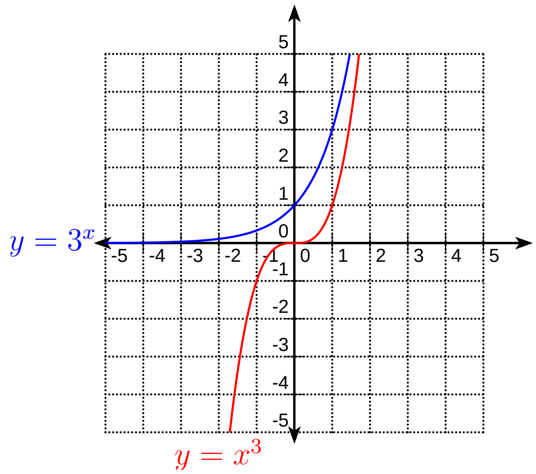
- $c^r c^s = c^{r+s}$
- $\dfrac{c^r}{c^s} = c^{r-s}$, $(c \neq 0)$
- $c^{-r} = \dfrac{1}{c^r}$, $(c \neq 0)$
- $(c^r)^s = c^{rs}$
- $c^{1/s} = \sqrt[s]{c}$
- $c^{r/s} = \sqrt[s]{c^r} = \left(\sqrt[s]{c}\right)^r$

10.2 Exponential Functions

Example 10.1 (Power Functions vs. Exponential Functions)

- Sketch the graphs of $y = P(x) = x^2$ and $y = E(x) = 2^x$ on the same graph.
- Sketch the graphs of $y = P(x) = x^3$ and $y = E(x) = 3^x$ on the same graph.





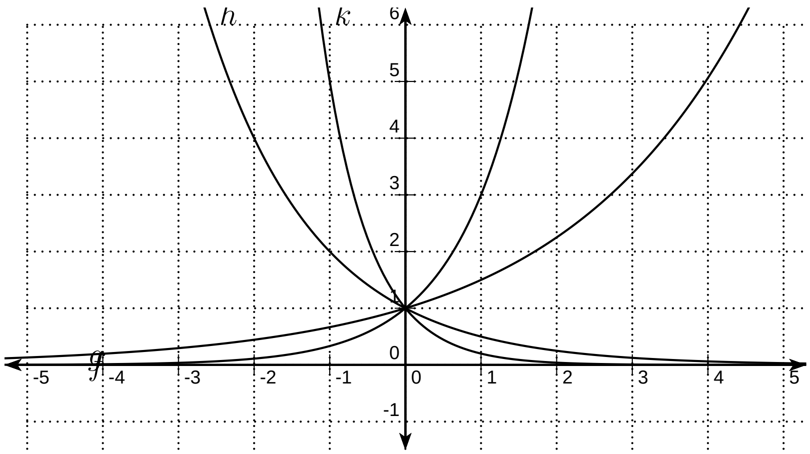
In the previous example, both of the P functions are *power functions*, and both of the E functions are *exponential functions*.

- What are the characteristics of a power function?
  - Has shape  or 
  - x-intercept and y-intercept is (0,0)
- What are the characteristics of an exponential function?
  - always positive
  - no x-intercept
  - y-intercept is (0,1)

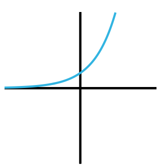
We can see that power functions are very different than exponential functions, so we should expect to treat them in very different ways. Solving a power equation is very different than solving an exponential equation. Finding the inverse function (if there is one) of a power function is very different than finding an inverse function of an exponential function. Do not be confused because both types of functions have exponents. **It matters if the variable is in the base or in the exponent.**

10.2.1 The Graphs of Exponential Functions

Example 10.2 (Exponential Graphs)
The graphs of $y = f(x) = 3^x$, $y = g(x) = 1.5^x$, $y = h(x) = 0.5^x$ and $y = k(x) = 0.2^x$ are drawn below for you. Label the graphs with their function names. Compare and contrast the graphs.

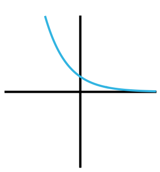


What are some characteristics of the graph of $y = f(x) = a^x$ if $a > 1$?



- y-intercept (1,0)
- no x-intercept
- always positive
- always increasing
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- one to one

What are some characteristics of the graph of $y = f(x) = a^x$ if $0 < a < 1$?



- y-intercept (1,0)
- no x-intercept
- always positive
- always decreasing
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- one to one

What happens if you try to graph $y = b(x) = (-2)^x$?
Since $(-2)^x$ is not defined for most values of x, the graph is a bunch of isolated points.

What happens if you try to graph $y = c(x) = 1^x$?
Since $c(x) = 1^x = 1$ for all x, then the graph of $c(x)$ is the line $y = 1$.

What happens if you try to graph $y = d(x) = 0^x$?
The domain of $d(x)$ is $(0, \infty)$ and $d(x) = 0^x = 0$ for $x > 0$. So, the graph of $d(x)$ is the positive x-axis.

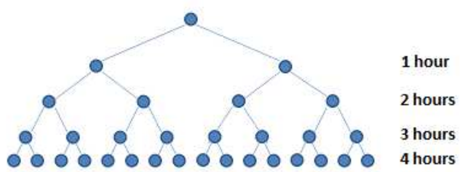
10.2.2 Understanding Exponential Functions

Definition 10.3 (Exponential Functions)
Let a be a positive number that is not equal to one. The **exponential function** with base a is a function that is equivalent to $f(x) = a^x$.

NOTE: Your textbook does not tell you that $a \neq 1$. However, because this function behaves so differently when $a = 1$, most textbooks do not call $g(x) = 1^x$ an exponential function. In this course, we will follow the convention that $g(x) = 1^x$ is **NOT** an exponential function.

Notice that $b(x)$, $c(x)$, and $d(x)$ in Example 10.2 are not exponential functions.

Example 10.4 (Understanding Exponential Growth)
Suppose that you place a bacterium in a jar. Each bacterium divides into 2 bacteria every hour.



- How many bacteria are in the jar after 2 hours?
  4
- How many bacteria are in the jar after 4 hours?
  16
- How many bacteria are in the jar after 9 hours?
  $1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^9 = 512$
- Write a function to express the total number, P, of bacteria are in the jar after t hours?
  To find the number of bacteria in hour $t + 1$, multiply the number of bacteria in hour t by 2. So,
  $$P(t) = 2^t$$
- At the 30 hour mark, the jar was completely full. How many bacteria were in the jar at this time?
  $P(30) = 2^{30}$
- When was the jar half full?
  At hour 29.

Lots of quantities grow by a certain multiple. For example, a bacteria population model may claim that the bacteria population is doubling every hour (as in the previous example). These situations yield what is known as **exponential growth models**.

Definition 10.5 (Exponential Growth)
If a quantity, P, can be modeled by a function of the form

$$P(t) = P_0 a^t,$$

where $a > 1$ and t represents time, then P is said to **grow exponentially**.

Notice that P_0 is the initial amount of the quantity because $P(0) = P_0 a^0 = P_0$.

Example 10.6
A bacteria culture starts out with 1000 bacteria and doubles every 3 hours. How many bacteria will there be after 5 hours? First, we have to use that $P(3) = 2000$ to find the constant a.

$$\begin{aligned} P(t) &= 1000a^t \\ P(3) &= 1000a^3 \\ 2000 &= 1000a^3 \\ 2 &= a^3 \\ 2^{1/3} &= a \end{aligned}$$

Since $a = 2^{1/3}$ then,

$$\begin{aligned} P(t) &= 1000\left(2^{1/3}\right)^t \\ P(5) &= 1000\left(2^{1/3}\right)^5 \\ &\approx 3174.8 \text{bacteria} \end{aligned}$$

There is a similar phenomenon called **exponential decay**. This occurs when $0 < a < 1$.

Definition 10.7 (Exponential Decay)
If a quantity, Q, can be modeled by a function of the form

$$Q(t) = Q_0 a^t$$

where $0 < a < 1$ and t represents time, then Q is said to **decay exponentially**.

Exponential growth and exponential decay are, for all practical purposes, the same idea.

Example 10.8 (Radioactive Decay)
The half life of Actinium-225 (Ac-225) is 10 days. How much of a 30-gram sample of Ac-225 is left after one year? First, we have to use that $Q(10) = 15$ to find the constant a.

$$\begin{aligned} Q(t) &= 30a^t \\ P(10) &= 30a^{10} \\ 15 &= 30a^{10} \\ \frac{1}{2} &= a^{10} \\ \left(\frac{1}{2}\right)^{1/10} &= a \end{aligned}$$

$$\begin{aligned} Q(t) &= 30\left(\left(\frac{1}{2}\right)^{1/10}\right)^t \\ Q(t) &= 30\left(\frac{1}{2}\right)^{t/10} \\ Q(365) &= 30\left(\frac{1}{2}\right)^{365/10} \\ &\approx 3.087 \times 10^{-10} \text{grams} \end{aligned}$$

Example 10.9
JB Outlet Store is having a sale on tents. Every day a tent does not sell, its price is marked down 20%. If the price of a tent is \$100 on Sunday, what is the price of the tent on Friday if it has not sold?

If the tent is marked down by 20\$, then its new price is 80% of the original price. So we have,

Day	Price of Tent
Sunday	\$100
Monday	$0.80(\$100) = \$80$
Tuesday	$0.8(\$80) = 0.8(0.8(\$100)) = (0.8)^2(\$100) = \$64$
Wednesday	$0.8(\$64) = 0.8\left((0.8)^2(\$100)\right) = (0.8)^3(\$100) = \$51.2$
Thursday	$0.8(\$51.2) = 0.8\left((0.8)^3(\$100)\right) = (0.8)^4(\$100) = \$40.96$
Friday	$0.8(\$40.96) = 0.8\left((0.8)^4(\$100)\right) = (0.8)^5(\$100) \approx \$32.77$

On Friday, the tent will cost $\$100(0.8)^5 = \$32.77$

10.3 Compound Interest

In Definition 10.5 notice that a is the factor by which the quantity changes when t increases by one unit. This growth could also be described as a certain percentage rate which is compounded. For example, a population model may claim that the population of the earth is increasing by $p\%$ per year or an investment may grow by $p\%$ per year. When the quantity is increasing by the rate r (as a decimal) for every unit of time, then $a = 1 + r$, so that the model can be written as

$$P(t) = P_0(1 + r)^t.$$

A special case of exponential growth that you are sure to run into is compound interest. When the interest is compounded yearly, there is no ambiguity. However, many times interest is compounded semiannually, quarterly, monthly, or even weekly. The interest rate is still given as an *annual* interest and time is usually given in years. In these cases, we must introduce another variable n which is the number of times per year the interest is compounded. Then r is the annual interest rate and t is the number of years. Then we get

Proposition 10.10 (Compound Interest)
If a principal P_0 is invested at an interest rate r for a period of t years, then the amount $P(t)$ of the investment is given by:

$$P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt} \quad \textit{(if compounded } n \textit{ times per year)}$$

Example 10.11 (Understanding Exponential Growth)
Suppose you invest \$10,000 in an account that earns 5% interest compounded semiannually.

- Write a function that expresses the amount of money in the account after t years.
- How much money will you have in 8 years?

Note that $P_0 = 10000, n = 2$, and $r = 0.05$, so

$$P(t) = 10000\left(1 + \frac{0.05}{2}\right)^{2t}$$

After 8 years, you will have

$$P(8) = 10000\left(1 + \frac{0.05}{2}\right)^{2(8)} \approx \$14,845.06$$

Example 10.12 (Understanding Exponential Growth)
Suppose you invest \$12,000 in an account that earns 3% interest compounded quarterly. How much money will you have in 1 year?

Note that $P_0 = 12000, n = 4$, and $r = 0.03$, so

$$P(t) = 12000\left(1 + \frac{0.03}{4}\right)^{4t}$$

After 1 year, you will have

$$P(1) = 12000\left(1 + \frac{0.03}{4}\right)^{4(1)} \approx \$12,364.07$$

In mathematics, there are a few very special numbers. The numbers 1 and 0 are special because they are the multiplicative and additive identities, the number π is special because it is an irrational number that is indispensable when you discuss circles, and e is a special irrational number for a multitude of reasons that you will only begin to understand in Calculus.

It is important to note that the number e is a number like π is a number. It is not a variable. It is an irrational number. You never can have an exact value for e. The best you can hope to have is a decimal approximation. $e \approx 2.71828182845....$

At this stage in your mathematical career, we can use compound interest to begin to explain the value of e.

Example 10.13 (The number e)
Suppose that you invest \$1 at an annual interest rate of 100% compounded annually. How much money will you have after 1 year? Note that $P_0 = 1, n = 1$, and $r = 1$

$$P(1) = 1\left(1+\frac{1}{1}\right)^{1(1)} \approx \$2$$

Suppose that you invest \$1 at an annual interest rate of 100% compounded monthly. How much money will you have after 1 year? Note that $P_0 = 1, n = 12$, and $r = 1$

$$P(1) = 1\left(1+\frac{1}{12}\right)^{12(1)} \approx \$2.61$$

Suppose that you invest \$1 at an annual interest rate of 100% compounded daily. How much money will you have after 1 year? Note that $P_0 = 1, n = 365$, and $r = 1$

$$P(1) = 1\left(1+\frac{1}{365}\right)^{365(1)} \approx \$2.71$$

Suppose that you invest \$1 at an annual interest rate of 100% compounded every minute. How much money will you have after 1 year? Note that $P_0 = 1, n = 365(24)(60)$, and $r = 1$

$$P(1) = 1\left(1+\frac{1}{365(24)(60)}\right)^{365(24)(60)(1)} \approx \$2.72$$

What does the value $\left(1+\frac{1}{n}\right)^n$ seem to be approaching as n becomes large?

2.71828

Suppose that you invest \$1 at an annual interest rate of 100% that could be compounded continuously. How much money should you expect to have after 1 year?

\$2.72

In Calculus, you will see that as n becomes very large, the quantity $\left(1+\frac{1}{n}\right)^n$ approaches the value e. This fact can be used to justify the following formula for continuously compounded interest.

Proposition 10.14 (Continuous Compounding)
If P_0 dollars is invested at an annual interest rate r (as decimal), compounded continuously, then the value of the investment after t years is given by

$$P(t) = P_0 e^{rt}.$$

Example 10.15 (Continuously Compounded Interest)
Jake invests \$1000 at an annual interest rate of 4.6% compounded continuously. How much money will Jake have in 15 years?

Note that $P = 1000, r = 0.46$, so

$$P(t) = 1000e^{0.046t}$$

After 15 years, Jake will have

$$P(15) = 1000e^{0.46(15)} \approx \$1993.72$$

10.4 Logarithms

Exponential functions are one-to-one functions. Consequently, each exponential function has an inverse function. Why might you want to undo exponentiation? Suppose you want to solve the following equation.

$$10^x = 3$$

What is happening to x? exponentiated by 10.

How do we undo this? Taking the x^{th} root is not a reasonable solution. This would lead to:

$$10 = \sqrt[x]{3}$$
$$10 = 3^{\frac{1}{x}}$$

This is even worse than before. We now have $\frac{1}{x}$ as an exponent. What we need is something to pull x out of the exponent place and put it on the ground, in a manner of speaking. Logarithms are the answer. One Calculus teacher was fond of saying that **logarithms are "exponent pickers."**

Recall that the name of a function does not need to be a single letter. We have used f and g to mean lots of different functions. But some functions occur so regularly that it makes more sense to give them permanent names that are a bit more descriptive. This is the case with logarithms. Logarithms will have names like log, $\log_2$, $\log_3$, and ln. Because these are functions, they have inputs and these inputs are placed in parentheses next to the function name. For example $\log(x)$, is the output of the function named log when x is the input of the function.

10.4.1 Logarithms with base a

The logarithm with base a is the inverse function of $f(x) = a^x$. The name of the logarithm with base a function is $\log_a$ (said "log base a").

Recall that x and y trade places in inverse functions. This leads to the following definition for the logarithm with base a function.

Definition 10.16 (Logarithms with Base a)
Let x and y be real numbers with $x > 0$. Let a be a positive real number that is not equal to 1. Then

$$\log_a(x) = y \text{ if and only if } a^y = x.$$

In other words, the $\log_a(x)$ picks the exponent to which a must be raised to produce x.

There are a few special logarithms with special names. The logarithm with base 10 is most often called the **common logarithm** is written $\log(x)$. The logarithm with base e is most often called the **natural logarithm** and is written $\ln(x)$.

Example 10.17 (Exponential Notation and Logarithmic Notation)
Convert the exponential statement to a logarithmic statement.

$$5^3 = 125 \qquad \log_5(125) = 3$$

$$10^{-3} = \frac{1}{1,000} \qquad \log\left(\frac{1}{1000}\right) = -3$$

$$e^2 \approx 7.389 \qquad \ln(7.389) \approx 2$$

Example 10.18
Convert the logarithmic statement to an exponential statement.

$$\log_3(3^5) = 5 \qquad 3^5 = 3^5$$

$$\log(10,000) = 4 \qquad 10^4 = 10000$$

$$\ln(1) = 0 \qquad e^0 = 1$$

Example 10.19
Evaluate each of the following.

- $\log(100) = 2$ Think $10^? = 100$
- $\log(10^9) = 9$ Think $10^? = 10^9$
- $\log\left(\frac{1}{1000}\right) = \log(10^{-3}) = -3$ Think $10^? = \frac{1}{1000}$

- $\log(\sqrt[3]{100}) = \log(100^{1/3}) = \log\left((10^2)^{1/3}\right) = \log 10^{2/3} = \dfrac{2}{3}$

- $\log_2(16) = 4$

- $\ln(e^5) = 5$

- $\ln\left(\dfrac{1}{\sqrt{e}}\right) = \dfrac{-1}{2}$

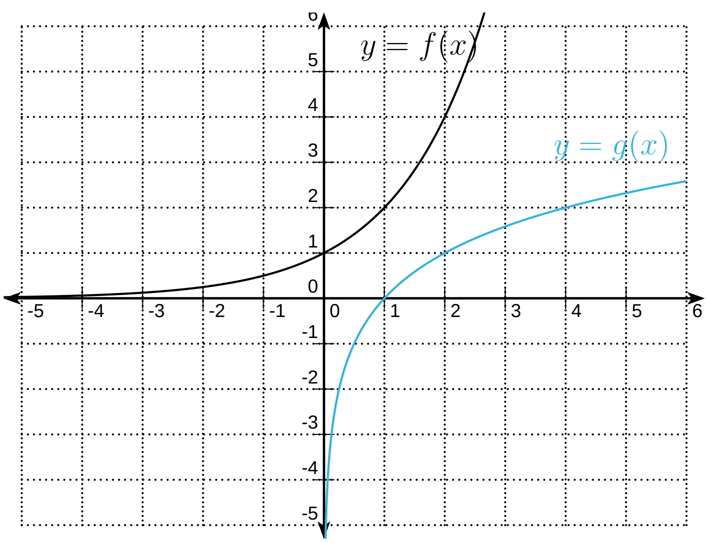
- $\log_3\left(\dfrac{1}{81}\right) = -4$

- $\log_{\frac{1}{2}}(16) = -4$

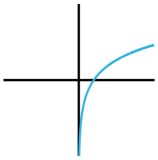
Example 10.20 (Logarithm with base 2 Graph)

- The graph of $y = f(x) = 2^x$ is drawn below. Sketch the graph of $y = g(x) = \log_2(x)$ on the same coordinate system.

- What is the domain of $g(x) = \log_2(x)$? What is the range of $g(x) = \log_2(x)$?

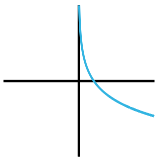


What are some characteristics of the graph of $y = f(x) = \log_a(x)$ if $a > 1$?



- x-intercept (1,0)
- no y-intercept
- always increasing
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- one to one

What are some characteristics of the graph of $y = f(x) = \log_a(x)$ if $0 < a < 1$?



- x-intercept (1,0)
- no y-intercept
- always decreasing
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- one to one

Notice that the input of a logarithm must be greater than 0. This is one more function where you must be careful when finding the domain.

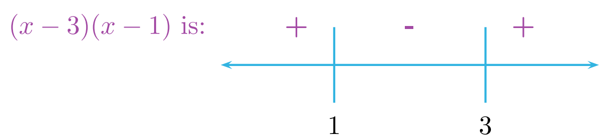
Example 10.21 (Logarithm Domain)
Find the domain of $f(x) = \log_7(2 - 5x)$.

$f(x)$ is defined when $2 - 5x > 0$, or $x < \dfrac{2}{5}$. That is the domain of $f(x)$ is $\left(-\infty, \dfrac{2}{5}\right)$.

Example 10.22 (Logarithm Domain)
Find the domain of $f(x) = \ln(x^2 - 4x + 3)$.

$f(x)$ is defined when $x^2 - 4x + 3 > 0$.

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x-3)(x-1) &= 0 \\ x - 3 = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = 3 \quad & \quad\quad\; x = 1 \end{aligned}$$



Thus, the domain of $g(x)$ is $(-\infty, 1) \cup (3, \infty)$.

10.4.2 Properties of Logarithms

Each property of logarithms is derived from the definition of the logarithm and/or a property of exponents.

Property 10.23

- $\log_a(1) = \underline{\ 0\ }$
- $\log_a(a) = \underline{\ 1\ }$
- $\log_a(a^x) = \underline{\ x\ }$
- $a^{\log_a(x)} = \underline{\ x\ }$

Notice that all properties can be stated in terms of the $\log_a(x)$ function since $\ln(x) = \log_e(x)$ and $\log(x) = \log_{10}(x)$

Example 10.24

- Simplify $e^{x\ln(2)}$.
  $e^{x\ln(2)} = e^{\ln(2)x} = \left(e^{\ln(2)}\right)^x = 2^x$
- Rewrite 5^x as e to a power.
  $5^x = \left(e^{\ln(5)}\right)^x = e^{x\ln(5)}$

Example 10.25
Evaluate $\log(10^5 * 10^3)$.

$\log(10^5 * 10^3) = \log(10^8) = 8$

Property 10.26 (Product Law for Logarithms)
For all $u > 0$ and $v > 0$

- $\log_a(uv) = \log_a(u) + \log_a(v)$

Proof:
Let $x = \log_a(u)$ and $y = \log_a(v)$. Then, $a^x = u$ and $a^y = v$. So,

$$uv = a^x a^y = a^{x+y}$$

Thus,

$$\log_a(uv) = \log_a\left(a^{x+y}\right) = x + y = \log_a(u) + \log_a(v)$$

$\square$

Property 10.27 (Quotient Law for Logarithms)
For all $u > 0$ and $v > 0$

- $\log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)$

Example 10.28
Use the properties of logarithms to express $\ln\left(\frac{xy}{z}\right)$ as a sum and or difference of three logarithms.

$$\ln\left(\frac{xy}{z}\right) = \ln(xy) - \ln(z) = \ln(x) + \ln(y) - \ln(z)$$

Example 10.29
Use the properties of logarithms to write the expression using the fewest number of logarithms possible.

$$\log(x^2+2) + \log(x) - \log(y) - \log(z)$$

$$\begin{aligned}
\log(x^2+2) + \log(x) - \log(y) - \log(z) &= \log(x^2+2) + \log(x) - [\log(y) + \log(z)] \\
&= \log\left((x^2+2)x\right) + \log(x) - \log(yz) \\
&= \log\left(\frac{(x^2+2)x}{yz}\right)
\end{aligned}$$

Property 10.30 (Power Law for Logarithms)
For all $u > 0$ and all k

- $\log_a(u^k) = k\log_a(u)$

Proof:
Let $x = \log_a(u)$. Then $a^x = u$. Thus,

$$u^k = (a^x)^k = a^{kx}$$

Hence,

$$\log_a(u^k) = \log_a\left(a^{kx}\right) = kx = k\log_a(u)$$

□

Example 10.31
Use the properties of logarithms to express $\log_5\left(\frac{x^3}{y\sqrt{z}}\right)$ in terms of $\log_5(x)$, $\log_5(y)$, and $\log_5(z)$.

$$\begin{aligned}
\log_5\left(\frac{x^3}{y\sqrt{z}}\right) &= \log_5 x^3 - \log_5\left(y\sqrt{z}\right) \\
&= 3\log_5(x) - \left[\log_5(y) + \log(\sqrt{z})\right] \\
&= 3\log_5(x) - \log_5(y) - \frac{1}{2}\log_5(z)
\end{aligned}$$

Example 10.32
Use the properties of logarithms to write the expression using the fewest number of logarithms possible.

$$\ln(x^2) - 2\ln(y) - 3\ln(z)$$

$$\begin{aligned}
\ln(x^2) - 2\ln(y) - 3\ln(z) &= \ln(x^2) - \ln(y^2) - \ln(z^3) \\
&= \ln(x^2) - \left[\ln(y^2) + \ln(z^3)\right] \\
&= \ln(x^2) - \ln(y^2z^3) = \ln\left(\frac{x^2}{y^2z^3}\right)
\end{aligned}$$

Property 10.33 (Change of Base) *If $a, b, x > 0$ and neither a nor b equals 1, then* $$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}.$$

Example 10.34 (Change of Base)
Use your calculator to approximate $\log_5(67)$.

$$\log_5(67) = \frac{\log(67)}{\log(5)} \approx 2.6125$$

10.5 Solving Exponential and Logarithmic Equations

Remember, "undoing" an operation means applying the inverse operation to both sides of an equation. The exponential function a^x and the logarithmic function $\log_a(x)$ are inverses of each other.

Example 10.35
Solve.

$$\log(x + 5) = 3$$

$$\begin{aligned}
x + 5 &= 10^3 \\
x + 5 &= 1000 \\
x &= 995
\end{aligned}$$

Example 10.36
Solve.

$$\log_8(x-5)+\log_8(x+2)=1$$

$$\begin{aligned}
\log_8(x-5)+\log_8(x+2) &= 1\\
\log_8\left((x-5)(x+2)\right) &= 1\\
(x-5)(x+2) &= 8^1\\
x^2-3x-10 &= 8\\
x^2-3x-18 &= 0\\
(x-6)(x+3) &= 0
\end{aligned}$$

So, $x=6$ and $x=-3$ are possible solutions.

However, we must check to make sure that the equation is defined for these solutions.

Remember, the logarithm is only defined for positive values.

$$\log_8(6-5)+\log_8(6+2)=\log_8(1)+\log_8(8)$$

Since both $\log_8(1)$ and $\log_8(8)$ are defined, then $x=6$ is a solution.

$$\log_8(-3-5)+\log_8(-3+2)=\log_8(-8)+\log_8(-1)$$

Since $\log_8(-8)$ and $\log_8(-1)$ are not defined, then $x=-3$ is not a solution.

The properties of logarithms only work when the input is positive. If you use them to solve an equation involving logarithms, **you must check your answer(s)**.

Example 10.37
Solve.

$$e^{x+2}=5$$

$$\begin{aligned}
x+2 &= \ln(5)\\
x &= \ln(5)-2\\
x &\approx -0.3906
\end{aligned}$$

Example 10.38
Solve.

$$\frac{2^x-7}{3}=-1$$

$$\begin{aligned}
2^x-7 &= -3\\
2^x &= 4\\
x &= \log_2(4)\\
x &= 2
\end{aligned}$$

Example 10.39

Solve.

$$2^{x-5} = 3^{2-2x}$$

$$\begin{aligned} 2^{x-5} &= 3^{2-2x} \\ \log_2\left(2^{x-5}\right) &= \log_2\left(3^{2-2x}\right) \\ x-5 &= (2-2x)\log_2(3) \\ x-5 &= 2\log_2(3) - 2x\log_2(3) \end{aligned}$$

$$\begin{aligned} x + 2x\log_2(3) &= 5 + \log_2(3) \\ x\,(1 + 2\log_2(3)) &= 5 + \log_2(3) \\ x &= \frac{5 + \log_2(3)}{1 + 2\log_2(3)} \approx 1.9592 \end{aligned}$$

Example 10.40

Joni invests \$1000 at an interest rate of 5% compounded monthly. When will the value of Joni's investment reach \$2500?

$$\begin{aligned} P(t) &= 1000\left(1 + \frac{0.05}{12}\right)^{12t} \\ 2500 &= 1000\left(1 + \frac{0.05}{12}\right)^{12t} \\ 2.5 &= \left(1 + \frac{0.05}{12}\right)^{12t} \end{aligned}$$

$$\begin{aligned} \ln(2.5) &= \ln\left(\left(1 + \frac{0.05}{12}\right)^{12t}\right) \\ \ln(2.5) &= 12t\ln\left(1 + \frac{0.05}{12}\right) \\ t &= \frac{\ln(2.5)}{12\ln\left(1 + \frac{0.05}{12}\right)} \approx 18.364\text{years} \end{aligned}$$

Example 10.41

A bacteria culture triples every 4 hours. How long until the culture doubles?

$$\begin{aligned} P(t) &= P_0 a^t \\ P(4) &= P_0 a^4 \\ 3P_0 &= P_0 a^4 \\ 3 &= a^4 \\ 3^{1/4} &= a \\ P(t) &= P_0\left(3^{1/4}\right)^t = P_0\,(3)^{t/4} \end{aligned}$$

$$\begin{aligned} P(t) &= P_0\,(3)^{t/4} \\ 2P_0 &= P_0\,(3)^{t/4} \\ 2 &= 3^{t/4} \\ \log_3 2 &= \frac{t}{4} \\ 4\log_3 2 &= t \\ 2.524\text{hours} &\approx t \end{aligned}$$