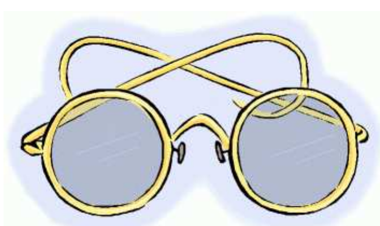
3 Solving Equations

Concepts:

- Number Lines
- The Definitions of Absolute Value
- Equivalent Equations
- Solving Equations with One Variable Type - The Algebraic Approach
- Solving Equations with a Variable in the Denominator - The Algebraic Approach
- Solving Power Equations - The Algebraic Approach
- Absolute Value Equations
- Solving Quadratic Equations - The Algebraic Approach
  - The Zero Product Property
  - The Quadratic Formula
  - Completing the Square
- Solving Quadratic Type Equations
- Other Types of Equations

(Sections 1.1-1.2 and Section 5.1A)

Algebra and Geometry do not live on different continents. They are not separate unrelated courses. They are more like two oceans. They have different characteristics, but their waters mingle. If you want to have the best chance of solving and understanding the solution of a problem, you will need both Algebra and Geometry. Very often, Geometry helps you to see the big picture while Algebra can help you work through nitty gritty details. If you want to understand your Calculus course, you will need to look at each problem through a geometric lens and an algebraic lens.



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Pictures are one of the best things about Geometry. If we are going to start to understand how to combine Algebra and Geometry, we need to find a way to draw a geometric picture of the numbers and variables that crop up in Algebra.

3.1 Pictures of Numbers: The Number Line

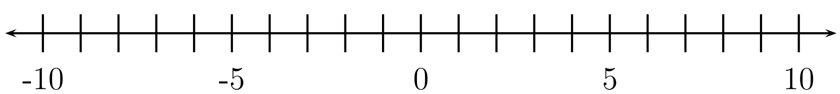
Every real number corresponds to a point on the number line. Every point on the number line corresponds to a real number.

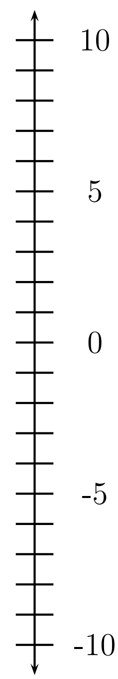
Number lines can be horizontal, vertical or any other direction. Most of the number lines we will use will be horizontal or vertical.

Traditionally, a smaller number appears to the left of a larger number on a horizontal number line. Traditionally, a smaller number appears below a larger number on a vertical number line.

Some things to know about the pictures you can draw with number lines:

1. Points that are shaded correspond to numbers that you want to include.
2. Points that are not shaded correspond to numbers that you do not want to include.
3. •, [, or] means that you include the number.
4. ○, (,) means that you do not include the number.

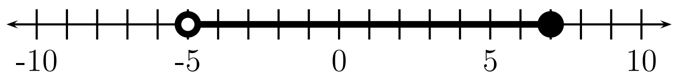




Your textbook allows you to use parentheses and brackets in the pictures that you draw on number lines. You are welcome to do this, but you also need to know the appropriate use for • and ○ on the number line. These are more useful when we move into higher dimensions, and your instructor is likely to use them in class and on exams.

3.2 Algebraic Notation for Number Line Pictures: Interval Notation

Example 3.1
Find the interval that corresponds to the graph.



A. $(7, -5]$

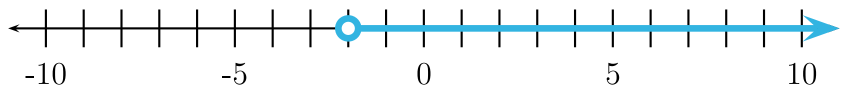
B. $[-5, 7)$

C. $[7, -5)$

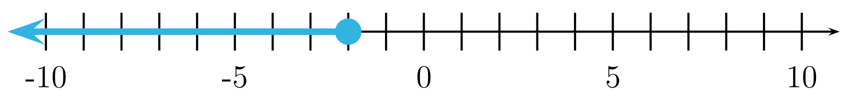
D. $\boxed{(-5, 7]}$

E. $(-5, 7)$

Example 3.2
Graph the interval $(-2, \infty)$ on a number line.

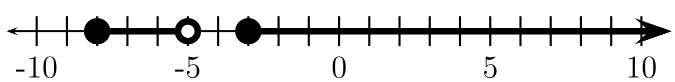


Example 3.3
Graph the interval $(-\infty, -2]$ on a number line.



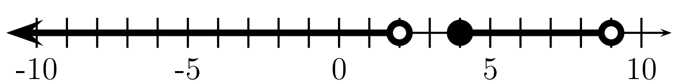
3.2.1 The Union Operator, $\cup$

If you need to include values that are in one interval OR another, we use the union operator. For example, the interval notation for



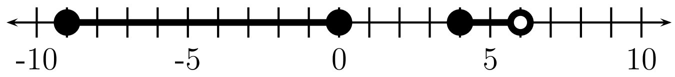
is $[-8, -5) \cup [-3, \infty)$.

Example 3.4 (Do you understand $\cup$?)
Write the interval notation that corresponds to the following graph?



$\boxed{(-\infty, 2) \cup [4, 9)}$

Example 3.5
Find the interval that corresponds to the graph.



A. $[0, -9] \cup (6, 4]$

B. $[0, -9] \cup [4, 6)$

C. $\boxed{[-9, 0] \cup [4, 6)}$

D. $(-10, 1) \cup (3, 6)$

E. $(-9, 0) \cup (4, 6]$

3.3 Distance on the Number Line: Absolute Value

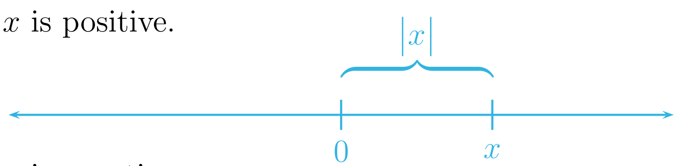
Undoubtedly, you have seen absolute values in other math classes. What you may not understand is why you have seen them. Do you know why we talk about absolute values?

The absolute value notation is shorthand for the distance between two points on a number line.

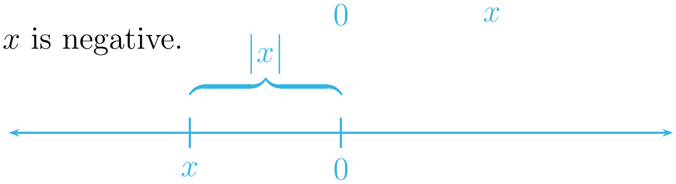
> **Definition 3.6 (Absolute Value - Geometric Defintion)**
> The **absolute value** of a number x, denoted $|x|$, is the distance between x and 0 on a number line.

Example 3.7
Draw a picture using the number line that represents the definition of $|x|$ when

1. x is positive.



2. x is negative.



Keeping in mind that algebra and geometry go hand in hand, let's try to convert this geometric idea to algebraic symbols.

If x is non-negative, then $|x| = x$.

If x is negative, then $|x| = -x$.

Definition 3.8 (Absolute Value - Algebraic Defintion)

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 3.9 (Do you understand Absolute Value?)

1. $|5.7| = 5.7$
2. $|-\pi| = \pi$
3. $|6-\pi| = 6-\pi$
4. $|2-\pi| = \pi - 2$

Several properties of the absolute value function are covered on page 10 of your textbook. **You are responsible** for reviewing these properties.

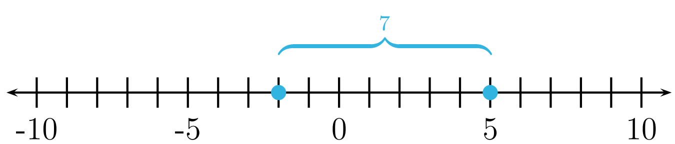
Two special properties that you may not recall are below.

Property 3.10
If c is a real number, then $\sqrt{c^2} = |c|$.

Property 3.11
If x and y are real numbers then $|x-y| = |y-x|$

So far, we have only used absolute values to find the distance between x and 0. Now let's find the distance between any two points on the number line.

Example 3.12
Find the distance between -2 and 5 on the number line. Can you write two different expressions for this distance using absolute value notation?



$\boxed{|5-(-2)| \text{ or } |-2-5|}$

Definition 3.13
The **distance between x and y on the number line** is

$$|x-y|$$

3.4 Solving Equations with One Variable Type - The Algebraic Approach

When two expressions are set equal to each other, the result is an **equation**. **Equations contain an equals sign. Expressions do not.** Equations may or may not contain variables. A **solution** to an equation is any substitution for the variables in an equation that results in a true mathematical statement.

Example 3.14 (Solutions to Equations)
Which of the following is a solution to $3 - 5x = 2(4 - x) + 1$?

- $\boxed{x = -2}$

$$\begin{aligned} 3 - 5(-2) &\stackrel{?}{=} 2(4 - (-2)) + 1 \\ 3 + 10 &\stackrel{?}{=} 2(6) + 1 \\ 13 &= 13 \checkmark \end{aligned}$$

So, $x = -2$ is a solution

- $x = 4$

$$\begin{aligned} 3 - 5(4) &\stackrel{?}{=} 2(4 - 4) + 1 \\ 3 - 20 &\stackrel{?}{=} 2(0) + 1 \\ -17 &\neq 1 \end{aligned}$$

So, $x = 4$ is not a solution

Consider the following equations:

$$x + 2 = 4$$

$$5x = 6 + 2x$$

$$3x^3 = 24$$

The only substitution for x that results in a true statement in each of these equations is $x = 2$. This means 2 is the only solution to each equation.

Equations that contain only one variable type can be solved by algebraic means. In this section, we will consider equations that have only one variable type.

Equation	Variable Types
$5x = 6 + 2x$	x
$3x^3 = 24$	x^3
$2\sqrt{x} = 7$	$\sqrt{x}$
$3x^2 = 2x + 1$	x, x^2

3.4.1 Equivalent Equations

For now, let us consider the pan-balance. The balance is level. Ideally, we want to perform operations on the equation that maintain the balance. Not only do we want to keep the balance, but we want to move toward a simple equation in which the solution is obvious. As we will see later, this is not always possible.

What operations can we perform that will always keep the pans in balance? In other words, what can be done to move from one equation to another equation so that both equations have the same solutions? This is a question about **equivalent equations**.

Definition 3.15
Two equations are **equivalent** if they have the same solutions.

When you are solving an equation, you want to move from one equation to an equivalent equation that is easier to solve.

For example,

$$2x + 5 = 3x - 1,$$
$$5 = x - 1, \text{ and}$$
$$6 = x$$

are equivalent equations since $x = 6$ is the only solution for ALL THREE equations.

Operations that Produce an Equivalent Equation:

1. Add or subtract the same number to both sides of the equation.
2. Add or subtract the same algebraic expression that is always defined to both sides of the equation.
3. Multiply or divide both sides of the equation by a **NONZERO** number.
4. Add zero to one side of the equation.
5. Multiply one side of the equation by 1.

Example 3.16 (Equivalent Equations)
In each of the following cases, decide if the action always, sometimes or never produces an equivalent equation. Justify your answer.

- Squaring both sides of an equation $\boxed{\text{Sometimes}}$

  Consider the equation $x = 2$. This has one solution, $x = 2$. Squaring both sides produces the equation $x^2 = 4$ which has two solutions, $x = 2$ and $x = -2$.

  Now, consider the equation $x = 0$. This has one solution, $x = 0$. Squaring both sides produces the equation $x^2 = 0$, which still only has the one solution $x = 0$.

- Adding x to both sides of an equation $\boxed{\text{Always}}$ This is always true by 2 in the above chart.

- Multiplying both sides of an equation by x $\boxed{\text{Sometimes}}$

  Consider the equation $x = 2$. This has one solution, $x = 2$. Multiplying both sides by x produces the equation $x^2 = 2x$, which has two solutions, $x = 2$ and $x = 0$.

  Now, consider the equation $x = 0$. This has one solution, $x = 0$. Multiplying both sides by x produces the equation $x^2 = 0x$, which still only has the one solution $x = 0$.

Ideally, we would like to keep equivalent equations as we move from one step to the next in the solution. This is not always possible. Sometimes you need to square both sides of the equation. Sometimes you need to multiply both sides of an equation by an algebraic expression instead of a number. These operations can produce **extraneous solutions**. This is why it is important to **CHECK YOUR SOLUTIONS**.

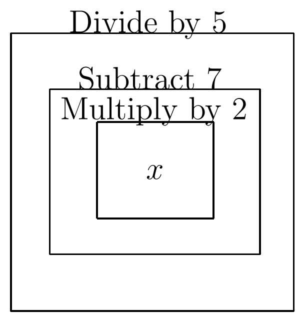
3.4.2 Unwrapping a Variable

If there is only one variable type in an equation, there are many ways that you can approach a problem. One possible strategy is to produce an equivalent equation in which the variable appears only one time in the equation. Then "unwrap" the variable until it is by itself on one side of the equation. Think about what happens to that variable. Unwrap one operation at a time by performing the *inverse* operation.

For example, suppose we want to solve

$$\frac{2x-7}{5} = 8.$$

What operations are being applied to x? In what order are these operations applied?



We need to work backwards through the list undoing one operation at a time.

To solve this equation we should do the following:

1. Multiply both sides by 5
2. Add 7 to both sides
3. Divide both sides by 2

Solve the equation.

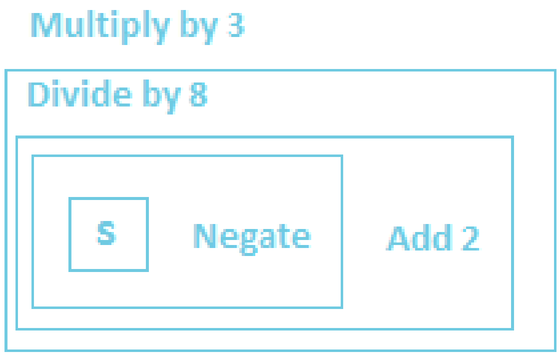
$$\begin{aligned} \frac{2x-7}{5} &= 8 \\ 2x-7 &= 40 \\ 2x &= 47 \\ x &= \frac{47}{2} \end{aligned}$$

Let's consider another example, suppose we want to solve

$$3\left(\frac{2-s}{8}\right)=5$$

What operations are being applied to s? In what order are these operations applied?

It is easier to solve equations if you think of subtraction as adding a negative number.



Example 3.17
Solve for s.

$$3\left(\frac{2-s}{8}\right)=5$$

$$\begin{aligned}\frac{2-s}{8} &= \frac{5}{3}\\ 2-s &= \frac{40}{3}\\ -s &= \frac{40}{3}-2\end{aligned}$$

$$\begin{aligned}-s &= \frac{34}{3}\\ &\boxed{s=\frac{-34}{3}}\end{aligned}$$

Example 3.18
Solve for s.

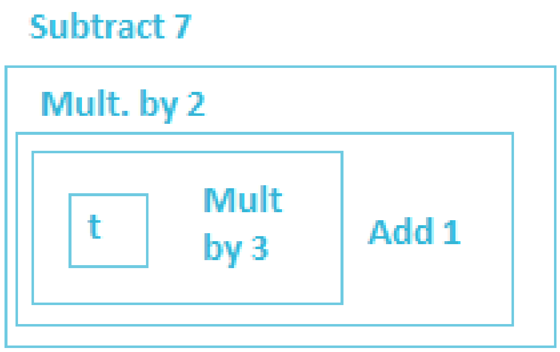
$$3\left(\frac{2-s}{8}\right)=\frac{s+6}{12}$$

$$\begin{aligned}\cancelto{3}{24}\cdot 3\left(\frac{2-s}{\cancel{8}}\right) &= \frac{s+6}{\cancel{12}}\cdot\cancelto{2}{24}\\ 9(2-s) &= 2(s+6)\\ 18-9s &= 2s+12\end{aligned}$$

$$\begin{aligned}6-9s &= 2s\\ 6 &= 11s\\ &\boxed{\frac{6}{11}=s}\end{aligned}$$

Example 3.19
Solve for t.

$$2(3t+1)-7=5$$



$$\begin{aligned} 2(3t+1)-7 &= 5 \\ 2(3t+1) &= 12 \\ 3t+1 &= 6 \\ 3t &= 5 \end{aligned}$$

$$\boxed{t=\frac{5}{3}}$$

Example 3.20
Solve for r.

$$C=2\pi r$$

$$\boxed{\frac{C}{2\pi}=r}$$

Example 3.21 (Concept Check)
To solve for b in the equation below, what should you do first?

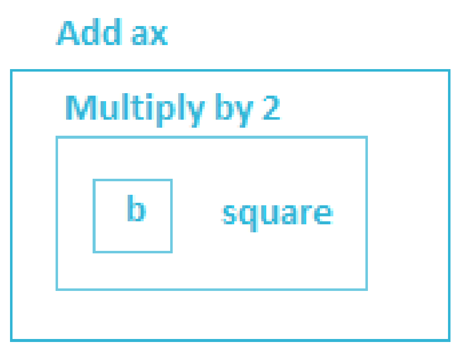
$$ax+b^2y=1$$

A. Divide both sides by a.

B. $\boxed{\text{Subtract } ax \text{ from both sides.}}$

C. Take the square root of both sides.

D. Divide both sides by y.



3.4.3 Solving Fractional Equations

When an equation has a variable in a denominator:

1. Find a common multiple for all denominators in the equation.
2. Multiply both sides of the equation by the common multiple.
3. Solve the new equation.

   Be CAREFUL! The new equation may not be equivalent to the original equation. You may find some **extraneous solutions** *when you solve the new equation.*

4. Check all of your solutions in the original equation. Keep only those solutions that are solutions of the original equation.

Example 3.22
Solve for y.

$$\frac{y}{y+1} = \frac{1}{y^2+y}$$

$$\begin{aligned}
\frac{y}{y+1} &= \frac{1}{y(y+1)} \\
\cancel{y(y+1)} \cdot \frac{y}{\cancel{y+1}} &= \frac{1}{\cancel{y}\cancel{(y+1)}} \cdot \cancel{y}\cancel{(y+1)} \\
y^2 &= 1 \\
y &= \pm 1
\end{aligned}$$

After checking both answers, we see $y = -1$ produces 0 in a denominator, so the only answer is $\boxed{y = 1}$

Example 3.23
Solve for u.

$$\frac{1}{F} = \frac{1}{v} + \frac{1}{u}$$

$$\begin{aligned}
uvF \cdot \frac{1}{F} &= \left(\frac{1}{v} + \frac{1}{u}\right) \cdot uvF \\
\frac{uvF}{F} &= \frac{uvF}{v} + \frac{uVF}{v} \\
uv &= uF + vF \\
uv - uF &= vF \\
u(v - F) &= vF
\end{aligned}$$

$$\boxed{u = \frac{vF}{v - F}}$$

3.4.4 Solving Power Equations

Solutions of Power Equations
The real solution(s) of $x^n = a$ is(are):

- $x = \sqrt[n]{a}$ if n is odd
- $x = \sqrt[n]{a}$ and $x = -\sqrt[n]{a}$ if n is even and $a \geq 0$

If n is even and $a < 0$, then $x^n = a$ does not have any real solutions.

Example 3.24
Solve.

(a) $x^4 = 10$

$$\boxed{x = \pm\sqrt[4]{10}}$$

(b) $\dfrac{x^3 + 5}{2} = 1$

$$\begin{aligned} x^3 + 5 &= 2 \\ x^3 &= -3 \end{aligned}$$

$$\boxed{x = \sqrt[3]{-3}}$$

(c) $3(x-4)^2 + 1 = 7$

$$\begin{aligned} 3(x-4)^2 &= 6 \\ (x-4)^2 &= 2 \\ x - 4 &= \pm\sqrt{2} \end{aligned}$$

$$\boxed{x = 4 \pm \sqrt{2}}$$

Example 3.25
Solve for r.

$$A = \pi r^2$$

$$\frac{A}{\pi} = r^2$$

$$\boxed{\pm\sqrt{\frac{A}{\pi}} = r}$$

3.4.5 A Return to Absolute Value

Number lines can be really insightful when working with absolute value equations. There are algebraic strategies for working with absolute value equations, but they appear to be some magic trick if you do not understand the geometry beneath the algebra.

Solutions of Absolute Value Equations
The real solution(s) of $\lvert x\rvert = a$ are $x = a$ and $x = -a$ if $a \geq 0$.

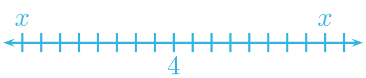
Example 3.26 (Distance Example)
Solve $\lvert x-3\rvert = 4$ geometrically. The distance from x to 3 is 4.

$\boxed{x = -1 \text{ or } x = 7}$

Example 3.27 (Another Distance Example)
Solve $\lvert 4-x\rvert + 3 = 11$.

$$\lvert 4-x\rvert = 8$$

The distance from x to 4 is 8.



$$\boxed{x = -4 \text{ or } x = 12}$$

Alternatively,

$$\lvert 4-x\rvert = 8$$

$$\begin{aligned} 4-x &= 8 \\ -x &= 4 \\ x &= -4 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4-x &= -8 \\ -x &= -12 \\ x &= 12 \end{aligned}$$

$$\boxed{x = -4 \text{ or } x = 12}$$

Example 3.28
Solve $\lvert 3x+2\rvert + 1 = 5$

$$\lvert 3x+2\rvert = 4$$

The distance from $3x$ to -2 is 4.

$$3x = -6 \text{ or } 3x = -2$$
$$\boxed{x = -2 \text{ or } x = \frac{2}{3}}$$

Alternatively,

$$\lvert 3x+2\rvert = 4$$

$$\begin{aligned} 3x+2 &= 4 \\ 3x &= 2 \\ x &= \frac{2}{3} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3x+2 &= -4 \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

$$\boxed{x = \frac{2}{3} \text{ or } x = -2}$$

3.5 Solving Equations with More than One Variable Type - The Algebraic Approach

When an equation has more than one variable type, there is often no way to find an equivalent equation which has only one variable. We must resort to other techniques. Nevertheless, some of these equations can still be solved algebraically, especially those equations that have "quadratic type". Others cannot be solved algebraically. In a few days, we will learn how to approximate the solutions of these equations geometrically.

3.5.1 Quadratic Equations

Definition 3.29
A **quadratic equation in** x is any equation that is equivalent to an equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$.

$2x^2 + 3x + 5 = 0$ is a quadratic equation in x.

$6u + 5u^2 = 2$ is a quadratic equation in u.

$\frac{4z^2 + 2}{5} = 7$ is a quadratic equation in z.

$2x + 3 = 0$ is not a quadratic equation.

$\frac{1}{x} + x^2 - 2 = 0$ is not a quadratic equation.

Property 3.30 (Zero Product Property)
If $AB = 0$ *then* $A = 0$ *or* $B = 0$

Example 3.31
Use the Zero Product Property to solve $x^2 - 9x = -20$.

$$\begin{aligned} x^2 - 9x &= -20 \\ x^2 - 9x + 20 &= 0 \\ (x-4)(x-5) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad x - 5 = 0 \end{aligned}$$

$$\boxed{x = 4 \text{ or } x = 5}$$

Example 3.32
Use the Zero Product Property to solve $x^2 + 5x - 6 = 0$.

$$\begin{aligned} x^2 + 5x - 6 &= 0 \\ (x+6)(x-1) &= 0 \\ x + 6 = 0 &\quad \text{or} \quad x - 1 = 0 \end{aligned}$$

$$\boxed{x = -6 \text{ or } x = 1}$$

Example 3.33
Use the Zero Product Property to solve $3x^2 - 5x + 2 = 0$.

$$\begin{aligned} 3x^2 - 5x + 2 &= 0 \\ (3x-2)(x-1) &= 0 \\ 3x - 2 = 0 &\quad \text{or} \quad x - 1 = 0 \\ 3x = 2 & \end{aligned}$$

$$\boxed{x = \frac{2}{3} \text{ or } x = 1}$$

While using the Zero Product Property is a nice method of solving a quadratic equation, it is not always easy (or possible) to factor a quadratic expression. We must use other means when this is the case.

Example 3.34 (A Factoring Example)
Factor the following expressions.

- $x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$
- $x^2 - 8x + 16 = (x-4)(x-4) = (x-4)^2$
- $x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)\left(x - \frac{7}{2}\right) = \left(x - \frac{7}{2}\right)^2$

Example 3.35 (Completing The Square)
Fill in the blank so that the following will factor as a perfect square.

$$x^2 - 12x + \underline{\hspace{3cm}}$$

(a) 24
(b) 6
(c) -6
(d) $\boxed{36}$
(e) -36
(f) 144

The idea behind completing the square is to move from an equation that has two variable types to an equation with only one variable type. We can solve equations with only one variable type by unwrapping the variable.

Example 3.36 (Completing The Square)
Solve $x^2 + 10x + 4 = 0$ by completing the square.

$$\begin{aligned} x^2 + 10x + 25 - 25 + 4 &= 0 \\ (x^2 + 10x + 25) - 25 + 4 &= 0 \\ (x+5)^2 - 21 &= 0 \\ (x+5)^2 &= 21 \\ x + 5 &= \pm\sqrt{21} \end{aligned}$$

$$\boxed{x = -5 \pm \sqrt{21}}$$

Example 3.37 (Completing The Square)
Solve $x^2 - 6x + 11 = 0$.

$$\begin{aligned} x^2 - 6x + 9 - 9 + 11 &= 0 \\ (x^2 - 6x + 9) - 9 + 11 &= 0 \\ (x-3)^2 + 2 &= 0 \\ (x-3)^2 &= -2 \end{aligned}$$

No solution.

Example 3.38 (Completing the Square)
Solve $2x^2 - 8x + 1 = 0$ by completing the square.

$$\begin{aligned} 2x^2 - 8x + 1 &= 0 \\ 2(x^2 - 4x) + 1 &= 0 \\ 2(x^2 - 4x + 4) - 2 \cdot 4 + 1 &= 0 \\ 2(x-2)^2 - 7 &= 0 \\ 2(x-2)^2 &= 7 \end{aligned}$$

$$\nearrow$$

$$\begin{aligned} (x-2)^2 &= \frac{7}{2} \\ x - 2 &= \pm\sqrt{\frac{7}{2}} \end{aligned}$$

$$\boxed{x = 2 \pm \sqrt{\frac{7}{2}} = 2 \pm \frac{\sqrt{14}}{2}}$$

Each time you complete the square, you are going through the exact same process. You could start with a generic quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) and complete the square with it. (This is done for you in your textbook.) Upon completing the square, you produce a formula for the solution(s).

Theorem 3.39 (The Quadratic Formula)
The solutions of $ax^2 + bx + c = 0$ ($a \neq 0$) are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Although memorizing is not the best strategy in mathematics, your life will be easier if you memorize the quadratic formula.

Example 3.40
Use the Quadratic Formula to solve $2x^2 - 8x + 1 = 0$.

Note that $a = 2$, $b = -8$, and $c = 1$. Thus, by the quadratic formula,

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{8 \pm \sqrt{56}}{4} = 2 \pm \frac{\sqrt{14}}{2}$$

Example 3.41
Use the Quadratic Formula to solve $3x^2 - 5x + 2 = 0$.

Note that $a = 3$, $b = -5$, and $c = 2$. Thus, by the quadratic formula

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \frac{5 \pm \sqrt{1}}{6} = 1, \frac{2}{3}$$

Compare this example to Example 3.33

Definition 3.42
In the quadratic formula, the expression $b^2 - 4ac$ is called the **discriminant**.

Theorem 3.43 (Number of Real Solutions of a Quadratic Equation)
If the discriminant of a quadratic equation is positive, the equation has two solutions. If it is zero, the equation has one solution. If it is negative, the equation does not have any real solutions.

Example 3.44 (The Discriminant)
How many real solutions does each equation have?

(a) $3x^2 + 2x + 5 = 0$
Since the discriminant $b^2 - 4ac = 2^2 - 4(3)(5) = -56 < 0$, then the equation has no real solutions.

(b) $x^2 + 5x = 7$
Since the discriminant $b^2 - 4ac = 5^2 - 4(1)(-7) = 53 > 0$, then the equation has two real solutions.

(c) $2x^2 = 12x - 18$ Rewrite as $2x^2 - 12x - 18$
Since the discriminant $b^2 - 4ac = (-12)^2 - 4(2)(18) = 0$, then the equation has one real solution.

3.5.2 Quadratic Type Equations

Some equations have the form $au^2 + bu + c = 0$ where u is an algebraic expression. We call these equations **quadratic type equations.**

To solve quadratic type equations:

1. Look for an expression and its square.
2. Let u be the expression.
3. Substitute u for the expression and u^2 for the square of the expression. The only variable in the new equation should be u. None of the original variables should remain.
4. Solve the new equation for u.
5. In the solution of the new equation, substitute the original expression for u. This will contain the original variable.
6. Solve for the original variable.
7. CHECK YOUR SOLUTIONS!

Example 3.45
Solve for x.

$$x^4 - 2x^2 - 3 = 0$$

Notice that $(x^2)^2 = x^4$ and let $u = x^2$. Substitution yields:

$$\begin{aligned} u^2 - 2u - 3 &= 0 \\ (u-3)(u+1) &= 0 \\ u - 3 = 0 \quad &\text{or} \quad u + 1 = 0 \end{aligned} \qquad \nearrow \qquad \begin{array}{rcl} u = 3 & \text{or} & u = -1 \\ x^2 = 3 & & x^2 = -1 \\ x = \pm\sqrt{3} & & \text{No solutions} \end{array}$$

$$\boxed{x = \pm\sqrt{3}}$$

Example 3.46
Solve for t.

$$2t^{1/6} + 8 = t^{1/3}$$

Notice that $(t^{1/6})^2 = t^{2/6} = t^{1/3}$. Let $u = t^{1/6}$, then the equation can be rewritten as $2u + 8 = 0$

$$\begin{aligned} 2u + 8 &= u^2 \\ 0 &= u^2 - 2u - 8 \\ 0 &= (u-4)(u+2) \\ u - 4 = 0 \quad &\text{or} \quad u + 2 = 0 \end{aligned} \qquad \nearrow \qquad \begin{array}{rcl} u = 4 & \text{or} & u = -2 \\ t^{1/6} = 4 & & t^{1/6} = -2 \\ t = 4^6 & & \text{No solutions} \\ t = 4096 & & \end{array}$$

$$\boxed{t = 4096}$$

Example 3.47
Solve for z.

$$\frac{1}{(z+1)^2} - 3 = \frac{2}{z+1}$$

Notice $\left(\frac{1}{z+1}\right)^2 = \frac{1}{(z+1)^2}$. Let $u = \frac{1}{z+1}$, then the equation can be rewritten as $u^2 - 3 = 2u$

$$\begin{aligned} u^2 - 3 &= 2u \\ u^2 - 2u - 3 &= 0 \\ (u-3)(u+1) &= 0 \\ u - 3 = 0 \quad &\text{or} \quad u + 1 = 0 \\ u = 3 \quad & \quad\;\; u = -1 \end{aligned} \qquad \nearrow \qquad \begin{array}{rcl} \frac{1}{z+1} = 3 & \text{or} & \frac{1}{z+1} = -1 \\ 1 = 3(z+1) & & 1 = -1(z+1) \\ 1 = 3z + 3 & & 1 = -z - 1 \\ -2 = 3z & & 2 = -z \\ \frac{-2}{3} = z & & -2 = z \end{array}$$

$$\boxed{z = \frac{-2}{3} \text{ or } z = -2}$$

3.5.3 Other Types of Equations

There are other types of equations that you have the ability to solve but do not fit nicely into any of the categories in this section. First, let's see how the Zero Product Property is useful for more than just quadratic equations.

Example 3.48
Find all real solutions to the equation.

$$4x^4 = 16x^2$$

$$\begin{aligned} 4x^4 - 16x^2 &= 0 \\ 4x^2(x^2 - 4) &= 0 \end{aligned}$$

$$\begin{array}{rcl} 4x^2 = 0 & \text{or} & x^2 - 4 = 0 \\ x^2 = 0 & & x^2 = 4 \\ x = 0 & & x = \pm 2 \end{array}$$

$$\boxed{x = 0, \pm 2}$$

Example 3.49
Find all real solutions to the equation.

$$7x^3 + 3x^2 - 3 = 10x - 3x^3$$

$$\begin{aligned} 10x^3 + 3x^2 - 10x - 3 &= 0 \\ x^2(10x + 3) - (10x + 3) &= 0 \\ (10x + 3)(x^2 - 1) &= 0 \end{aligned}$$

$$\begin{array}{rcl} 10x + 3 = 0 & \text{or} & x^2 - 1 = 0 \\ 10x = -3 & & x^2 = 1 \\ x = \dfrac{-3}{10} & & x = \pm 1 \end{array}$$

$$\boxed{x = \frac{-3}{10}, 1, -1}$$

The last type of equation we will solve in this section is equations with radicals. Previously, we stated that squaring both sides of an equation does not necessarily produce an equivalent equation. However, it may be necessary for equations which involve radicals. This may create extraneous solutions. **If you square both sides of an equation, you must check your answers.**

Example 3.50
Find all real solutions to the equation.

$$\sqrt{1-t} = t+5$$

$$\begin{aligned} \left(\sqrt{1-t}\right)^2 &= (t+5)^2 \\ 1-t &= t^2+10t+25 \\ 0 &= t^2+11t+24 \\ 0 &= (t+3)(t+8) \\ t+3=0 \quad &\text{or} \quad t+8=0 \\ t=-3 \quad & \quad\;\; t=-8 \end{aligned}$$

Checking $t=-3$

$$\begin{aligned} \sqrt{1-(-3)} &\stackrel{?}{=} -3+5 \\ \sqrt{4} &\stackrel{?}{=} 2 \\ 2 &= 2\checkmark \end{aligned}$$

Thus, $t=-3$ is a solution.

Checking $t=-8$

$$\begin{aligned} \sqrt{1-(-8)} &\stackrel{?}{=} -8+5 \\ \sqrt{9} &\stackrel{?}{=} -3 \\ 3 &\neq -3 \end{aligned}$$

Thus, $t=-8$ is not a solution.

$\boxed{\text{The only solution is } t=-3.}$

Example 3.51
Find all real solutions to the equation.

$$2+\sqrt{a} = a$$

$$\begin{aligned} 2+\sqrt{a} &= a \\ \sqrt{a} &= a-2 \\ \left(\sqrt{a}\right)^2 &= (a-2)^2 \\ a &= a^2-4a+4 \\ 0 &= a^2-5a+4 \\ 0 &= (a-1)(a-4) \\ a-1=0 \quad &\text{or} \quad a-4=0 \\ a=1 \quad & \quad\;\; a=4 \end{aligned}$$

Checking $a=1$

$$\begin{aligned} 2+\sqrt{1} &\stackrel{?}{=} 1 \\ 2+1 &\stackrel{?}{=} 1 \\ 3 &\neq 1 \end{aligned}$$

Thus, $a=1$ is not a solution.

Checking $a=4$

$$\begin{aligned} 2+\sqrt{4} &\stackrel{?}{=} 4 \\ 2+2 &\stackrel{?}{=} 4 \\ 4 &= 4\checkmark \end{aligned}$$

Thus, $a=4$ is a solution.

$\boxed{\text{The only solution is } a=4.}$

In the next section of the notes, we will examine the two-dimensional pictures that correspond to equations. These can be much more insightful than the one-dimensional number line picture. In order to graph equations, we must understand the Cartesian Coordinate System.

> The only way to guarantee an **exact** solution is to solve the equation **algebraically**. Solutions that are obtained by **graphical** means are **approximations**.