4 The Cartesian Coordinate System - Pictures of Equations

Concepts:

- The Cartesian Coordinate System
- Graphs of Equations in Two Variables
- x-intercepts and y-intercepts
- Distance in Two Dimensions and the Pythagorean Theorem
- Equations of Circles
  - The Distance Formula and the Standard Form for an Equation of a Circle.
  - Writing Equations of Circles
  - Identifying Equations of Circles
- Midpoints
  - Finding Midpoints
  - Verifying that a Point Is the Midpoint of a Line Segment
- Steepness
- Rates of Change
- Lines
  - The Slope
  - The Slope as a Rate of Change
  - Linear Equations
  - Point-Slope Form
  - Vertical Lines
  - Horizontal Lines
  - Parallel Lines and Perpendicular Lines

(Sections 1.3-1.4)

In this section, we leap from the one-dimensional number line to the two-dimensional Cartesian Coordinate System. This leap will allow us to see pictures of equations that contain two variables (x and y, for example). We will also see how these pictures can be used to help us approximate solutions of equations that contain only one variable.

4.1 Your Personal Review

Undoubtedly, you have seen the Cartesian Coordinate System in previous classes. We will not review the basics of plotting points during lecture, but you should read section 1.3 in your textbook for a quick review. In particular, you need to be able to:

- Locate points on a Cartesian Coordinate System.
- Discuss the four quadrants of a Cartesian Coordinate System.
- Identify the x-axis, the y-axis, and the origin on a Cartesian Coordinate System.

Example 4.1 (A Review of the Cartesian Coordinate System)
Every point on the x-axis has:

A. an x-coordinate that equals 0.

B. a y-coordinate that equals 0.

C. Both (A) and (B).

4.2 Graphs of Equations with Two Variables

Definition 4.2
The graph of an equation in the variables x and y is the set of all points (a, b) such that $x = a$, $y = b$ is a solution to the equation.

- If a point is on the graph of an equation, then the point is a solution of the equation.
- If a point is a solution of an equation, then the point is on the graph of the equation.
- If a point is **not** on the graph of an equation, the the point is **not** a solution of the equation.
- If a point is **not** a solution of an equation, then the point is **not** on the graph of the equation.

Example 4.3

- Is $(2, 3)$ on the graph of $y = 3x + 5$?

$3 \stackrel{?}{=} 3 \cdot 2 + 5$

$3 \stackrel{?}{=} 6 + 5$

$3 \neq 11$

No $(2,3)$ is NOT on the graph of $y=3x+5$.

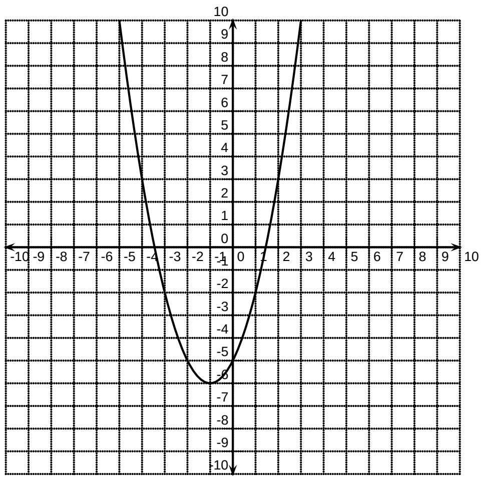
- Is $(1, 8)$ on the graph of $y = 3x + 5$?

$8 \stackrel{?}{=} 3(1) + 5$

$8 = 8$ ✓

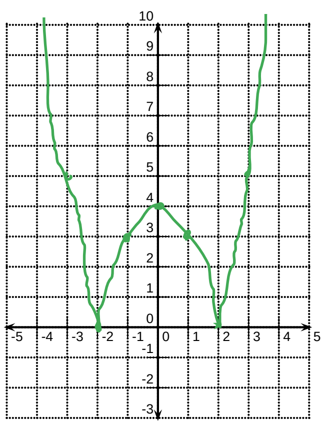
Yes $(1,8)$ is on the graph of $y=3x+5$.

Example 4.4
Explain why the graph below is **not** the graph of $y = x^2 + 2x + 3$.



$(0,3)$ is a solution to $y = x^2+2x+3$ but $(0,3)$ is not a point on the graph given. Therefore this is not the graph of $y=x^2+2x+3$.

Example 4.5
Sketch the graph of $y = |x^2 - 4|$ by making a table of values.



x	y
0	4
1	3
2	0
3	5
-1	3
-2	0
-3	5

Definition 4.6 (x-intercepts and y-intercepts)
If a graph intersects the x-axis at the point $(a, 0)$, then a is called an x-intercept of the graph.
If a graph intersects the y-axis at the point $(0, b)$, then b is called a y-intercept of the graph.

Notice that the y value equals zero at an x-intercept and the x-value equals zero at a y-intercept.

Example 4.7
Find the intercepts of the graph of $x = y^2 + 2y - 3$

x-int: $y=0$ $\quad x = 0^2 + 2\cdot 0 - 3 = -3$ $\quad (-3, 0)$ x-int

y-int: $x=0$ $\quad 0 = y^2 + 2y - 3$

$0 = (y+3)(y-1)$

$y+3=0 \quad y-1=0$

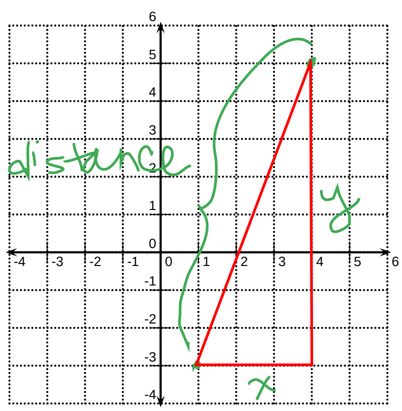
$y=-3 \quad y=1$

$(0,-3)$ $(0,1)$ y-int

4.3 Distance

When we were working with the one-dimensional number line, we saw the relationship between distance and the absolute value. In the two-dimensional Cartesian Coordinate System, we will use the Pythagorean Theorem to discuss distance. We will also see the relationship between the distance formula and the equations for circles.

Example 4.8
Use the Pythagorean Theorem to find the distance between the points $(4, 5)$ and $(1, -3)$.



By Pythagorean Theorem,

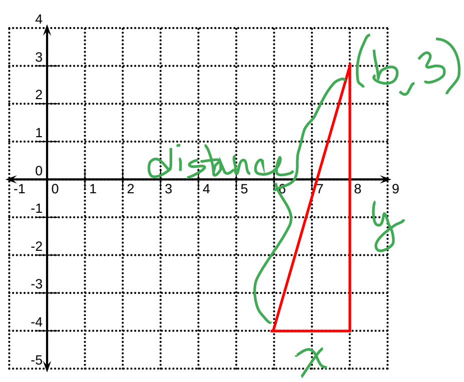
$x^2 + y^2 = \text{distance}^2$

$3^2 + 8^2 = d^2$

$73 = d^2$

$\sqrt{73} = d$

Example 4.9
Use the Pythagorean Theorem to find the distance between the points $(6, -4)$ and $(b, 3)$.



Note, b could be anything.

$y = 7$

$x = b - 3$

$x^2 + y^2 = \text{distance}^2$

$(b-3)^2 + 7^2 = \text{distance}^2$

$\text{distance} = \sqrt{(b-3)^2 + 49}$

Theorem 4.10 (The Distance Formula)
The distance between the points (x_1, y_1) *and* (x_2, y_2) *is*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The Distance Formula is just the Pythagorean Theorem in disguise.

Example 4.11
Find the distance between the points $(-1, 2)$ and $(7, 4)$.

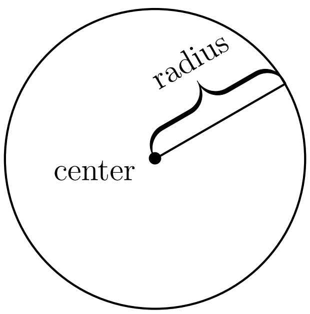
distance $= \sqrt{(7-(-1))^2 + (4-2)^2}$

$= \sqrt{68}$

4.3.1 Equations of Circles

Recall the definition of a circle.

Definition 4.12
A **circle** is the set of all points that are a fixed distance r from a specified point called the **center** of the circle. The distance r is called the **radius** of the circle.



Since the definition of a circle is based on a distance, we can use the distance formula to find the equation of a circle.

Example 4.13
Find an equation for the circle with center $(-2, 5)$ and radius 4.

Let (x, y) be any point on the circle.
Then the distance from (x, y) to $(-2, 5)$ is 4.

$\sqrt{(x-(-2))^2 + (y-5)^2} = 4$

$\boxed{(x+2)^2 + (y-5)^2 = 16}$

Theorem 4.14 (Circle Equations-Standard Form)
An equation for a circle with center (h, k) and radius r is equivalent to

$$(x-h)^2 + (y-k)^2 = r^2.$$

Example 4.15
Find an equation for the circle with center $(2, -1)$ that passes through the point $(4, -6)$.

radius is $r = \sqrt{(2-4)^2 + (-1-(-6))^2} = \sqrt{4+25} = \sqrt{29}$

Equation $\boxed{(x-2)^2 + (y+1)^2 = 29}$

Example 4.16 (Do you understand circle equations?)
What is the center of the circle whose equation in standard form is $(x-3)^2 + y^2 = 25$?

$(x-3)^2 + (y-0)^2 = 25$

(a) $(3, 5)$

(b) $(0, 3)$

(c) $(3, 0)$

(d) $(-3, 0)$

(e) The graph of this equation is not a circle.

Example 4.17
Is the graph of $x^2 + 10x + y^2 - 6y + 32 = 0$ a circle? If so, find its center and radius.

Complete the Square!

$x^2 + 10x + 25 - 25 + y^2 - 6y + 9 - 9 + 32 = 0$

$(x+5)^2 + (y-3)^2 - 2 = 0$

$(x+5)^2 + (y-3)^2 = 2 \leftarrow$ Standard Form for Circle

Center $(-5, 3)$ radius $\sqrt{2}$

Example 4.18
Is the graph of $x^2 + 4x + y^2 - 2y + 40 = 0$ a circle? If so, find its center and radius.

$x^2 + 4x + 4 + y^2 - 2y + 1 - 4 - 1 + 40 = 0$

$(x+2)^2 + (y-1)^2 + 35 = 0$

$(x+2)^2 + (y-1)^2 = \underbrace{-35}$

Not positive so can't be r^2

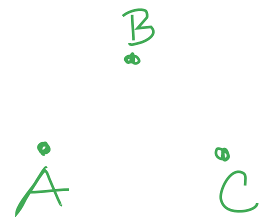
Not an equation for a circle.

4.3.2 Midpoints

Definition 4.19 (Midpoint)
The **midpoint** of the line segment AB is the point on the line segment that is equidistant from A and B.

Example 4.20
Suppose that the distance from A to B equals the distance from B to C. Is it necessarily true that B is the midpoint of AC? Why or why not?

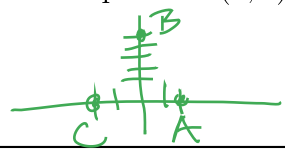
Not necessarily. Think about



What extra piece of information do you need?

The distance from A to C equals the distance from A to B plus the distance from B to C.

Example 4.21
Consider the points $A(2,0)$, $B(0,5)$, and $C(-2,0)$. Is B the midpoint of AC?



No.

Theorem 4.22
The midpoint of the line segment from (x_1, y_1) *to* (x_2, y_2) *is*

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Proof. This proof is left as an exercise. $\square$

Example 4.23 (Circles and Diameters)
A diameter of a circle has endpoints $(2,5)$ and $(4,-1)$. Find an equation for the circle.

To write down an equation for a circle in standard form, you need the radius & center.

center: $\left(\frac{2+4}{2}, \frac{5+(-1)}{2}\right) = (3,2)$

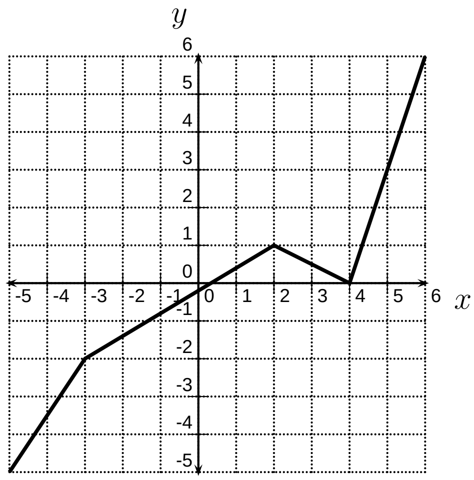
radius: $r = \sqrt{(3-2)^2 + (2-5)^2} = \sqrt{10}$

$(x-3)^2 + (y-2)^2 = 10$

4.4 Steepness, Lines, and Rates of Change

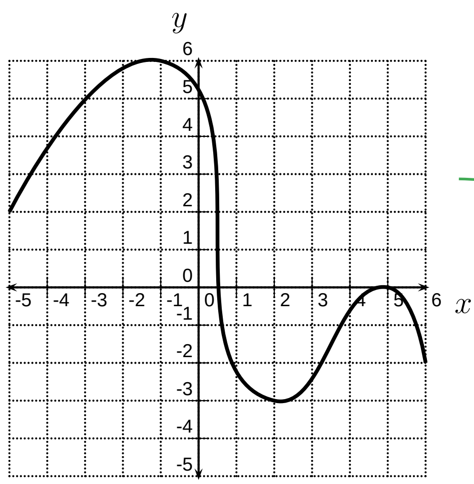
The distance formula allows us to measure the length of a line segment in two dimensions. Distance or length is just one of the useful quantities we can measure in two dimensions. The steepness of a line segment is another useful measurement in two dimensions. How fast is a line segment rising or falling?

Example 4.24
Consider the curve shown below. Discuss the steepness of the curve.



The curve gets less steep, then starts going down then gets very steep.

Example 4.25
Consider the curve shown below. Discuss the steepness of the curve.



It's hard to talk about the steepness of this curve as it changes gradually. (The steepness of curves such as this is an important topic in calculus.)

Example 4.26
Suppose you want to draw a curve for which the steepness does not vary. What type of curve could you draw?

a line

We have been discussing steepness in very vague qualitative terms. If we want to be able to make more concrete comparisons, we need to be able to quantify, or measure, the steepness of a curve. You will need Calculus in order to measure the steepness of general curves like the curve shown in Example 4.25. Today we will quantify the steepness of lines because they are the simplest curves. Moreover, this measurement will lay a foundation for what you will do in Calculus.

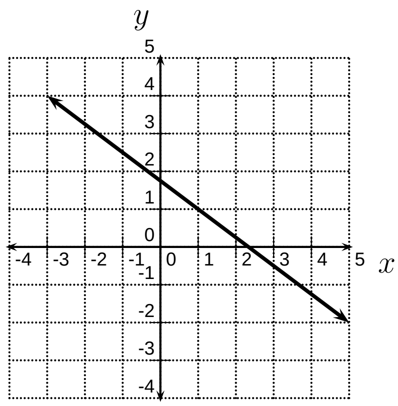
4.4.1 Quantifying the Steepness of a Line - Slope

The steepness of a line is determined by comparing the change in the vertical distance to the change in the horizontal distance. The ratio of the change in vertical distance to the change in horizontal distance is constant for a line. For a line, this ratio is called the **slope** of the line.

Definition 4.27
If $x_1 \neq x_2$, then the slope of the line through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 4.28
Find the slope of the line shown below.



(−3,4) and (5,−2) are points on the line.

$$m = \frac{4-(-2)}{-3-5} = \frac{6}{-8} = -\frac{3}{4}$$

Example 4.29
Find the slope of the line that passes through $(6, 2)$ and $(6, 5)$.

$$m = \frac{2-5}{6-6} = \frac{-3}{0} \text{ undefined}$$

($\Delta x = 0$ for every vertical line)

Example 4.30
Find the slope of the line that passes through $(1, 3)$ and $(-5, 3)$.

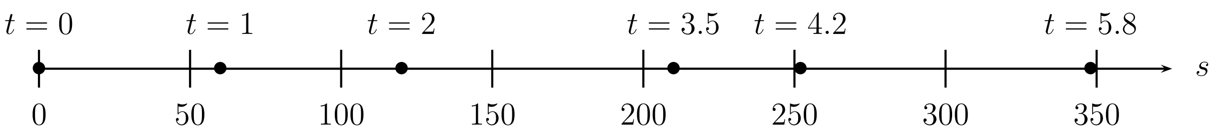
$$m = \frac{3-3}{1-(-5)} = \frac{0}{6} = 0$$

($\Delta y = 0$ for every horizontal line)

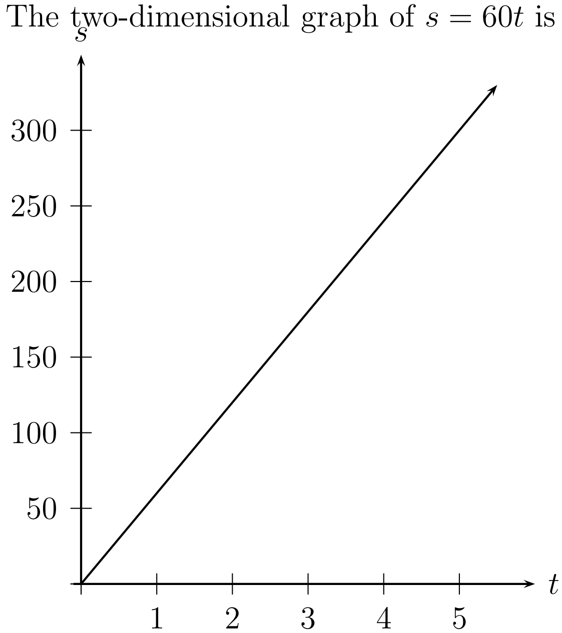
4.4.2 Rates of Change

The slope of a line is a **rate of change**. If the line is in terms of the variables x and y, then the slope of the line is the rate of change of y with respect to x. This means that the slope is the ratio of the change in y to the change in x. The word **"per"** is often associated with rates of change. Think about speed. Speed can be measured in miles **per** hour. Speed is a rate of change of distance with respect to time.

Example 4.31 (Rate of Change)
A particle is traveling along a straight line. Its position, s, at time t seconds is given by $s = 60t$ where s is measured in feet.



The two-dimensional graph of $s = 60t$ is shown below. This graph is linear.



1. What is the slope of this line?

   (0,0) & (1,60) are on line $\quad \frac{60-0}{1-0} = 60$

2. What are the units of the slope?

   feet per second

3. Express the slope of the line as a rate of change.

   60 feet per 1 second

4. What does the slope of this line tell us about the particle?

   The particle's position changes 60 feet every 1 second.

4.4.3 Equations of Lines

As we said before, the slope of a non-vertical line is constant. A line can be totally described by its slope and a point on the line. Why do we need a point? Why is the slope not sufficient?

You must place the line somewhere. same slope different lines.

Just as we used the distance formula to construct an equation for a circle, we will now use the slope formula to construct an equation for a non-vertical line.

Example 4.32
Find an equation for the line that passes through $(4, 3)$ and has slope -2.

Let (x, y) be any point on the line other than $(4,3)$. By the slope formula,

$$\frac{y-3}{x-4} = -2.$$

Every point on the line satisfies this equation except $(4,3)$. So we manipulate it to get

$$\boxed{y-3=-2(x-4)}$$

Example 4.33
Find an equation for the line that passes through (a, b) and has slope -2.

$$\frac{y-b}{x-a} = -2$$

$$\boxed{y-b=-2(x-a)}$$

Theorem 4.34 (Equations of Lines-Point Slope Formula)
The graph of

$$y - b = m(x - a)$$

is a line that passes through (a, b) and has slope m.

Theorem 4.35 (Equations of Lines-Horizontal and Vertical Lines)
The graph of $y = b$ is a horizontal line that passes through (a, b). The slope of a horizontal line is 0. The graph of $x = a$ is a vertical line that passes through (a, b). The slope of a vertical line is undefined.

Definition 4.36 (Linear Equations)
A **linear equation** in x and y is an equation that is equivalent to an equation of the form

$$Ax + By + C = 0$$

where A, B, and C are constants.

Example 4.37
Verify that $y - b = m(x - a)$ is a linear equation. What are the values of A, B, and C in the definition of a linear equation.

$y - b = mx - ma$

$\underbrace{-mx}_{A} + \underbrace{y}_{B} + \underbrace{ma - b}_{C} = 0$

$A = -m$

$B = 1$

$C = ma - b$

Example 4.38 (Concept Check)
Which of the following are linear equations?

A. $2x + 4y = 5$ B. $y - 3 = 4(x - 2)$ C. $y = \dfrac{2}{x+5}$ D. $x = 4y - 3$

E. $y = -\dfrac{2}{3}x$ F. $(x - 1)^2 + (y + 2)^2 = 3$

4.4.4 Miscellaneous Information About Lines

The following facts about lines are **IMPORTANT**:

- Slope-intercept form ($y = mx + b$) is **NOT** the only way to write an equation for a line. Sometimes it is not even the most useful way to write an equation for a line.
- Point-slope form is often more useful than slope-intercept form. **Learn it!**
- Slope-intercept form is useful if you need to determine the slope of a line or if you you are checking to see if two lines are the same.
- Parallel lines have the same slope.
- The product of the slopes of perpendicular lines is -1.
- The slope of vertical lines ($x = a$) is undefined.
- The slope of horizontal lines ($y = a$) is zero.
- To find the equation of a line, you need the slope and a point.

Example 4.39
Find an equation for the line that passes through $(-9, 5)$ and is parallel to the line whose equation is $y - 2 = \frac{7}{4}(x + 1)$.

Line I want
- goes through $(-9, 5)$

$m = \frac{7}{4}$

$\boxed{y - 5 = \frac{7}{4}(x + 9)}$

Line parallel
- $y - 2 = \frac{7}{4}(x + 1)$

Slope $\frac{7}{4}$

Slopes are same

Example 4.40
Find an equation for the line that is perpendicular to $2x + 3y - 5 = 0$ and passes through the point $(2, -1)$.

Line I want
- goes through $(2, -1)$

$m\left(-\frac{2}{3}\right) = -1$

$m = \frac{3}{2}$

$\boxed{y + 1 = \frac{3}{2}(x - 2)}$

Line perpendicular

$2x + 3y - 5 = 0$

$y = -\frac{2}{3}x + \frac{5}{3}$

Slope $-\frac{2}{3}$

Example 4.41
Find the slope of the line given by the equation $4x + 5y = 10$? What is its y-intercept? Graph the line.

y-int: $x = 0$

$4 \cdot 0 + 5y = 10$

$5y = 10$

$y = 2$

$(0, 2)$ y-int

Another point on line is $(5, -2)$