9 Functions

Concepts:

- The Definition of A Function
  - Identifying Graphs of Functions (Vertical Line Test)
- Function Notation
- Piecewise-defined Functions
  - Evaluating Piecewise-defined Functions
  - Sketching the Graph of a Piecewise-defined Functions
- The Domain of a Function
- Graphs of Functions
  - Interpreting Graphs of Functions
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- Average Rates of Change
  - Calculating the Average Rate of Change of a Function
  - Secant Lines
  - Difference Quotients
- Operations on Functions
- The Domain of a Composition of Functions.
- Graph Transformations
- One-to-one Functions and Inverse Functions
  - The Definition of a One-to-one Function
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  - Finding Formulas for Inverse Functions
  - Evaluating Inverse Functions
  - Graphs of Inverse Functions
  - The Round-Trip Theorem

(Chapter 3)

9.1 The Function Concept

Functions are very nice machines. You put something into the machine, the machine does some work, and the machine produces something predictable. For example, consider a Coke machine. If you put a dollar into the machine and press the Sprite button you expect the machine to produce a bottle of Sprite because it always produces a bottle of Sprite when you the press the Sprite button. The Coke machine is a function. The Sprite button always produces a bottle of Sprite, the Coke button always produces a Coke, and the Fresca button always produces a Fresca.

On the other hand, suppose that you decide to try something different. On Tuesday, you go to the Pepsi machine. You press the Mountain Dew button and the machine produces a Mountain Dew, as expected. On Wednesday, you press the Mountain Dew button again, but this time the machine produces a Pepsi. This is very unpredictable behavior. The Pepsi machine is **not** a function. The same input (the Mountain Dew) button produced two different outputs (Mountain Dew and Pepsi).

Definition 9.1 (Functions)
A **function** consists of:

- A **set of inputs** called the **domain**,
- A rule which assigns to each input exactly one output, and
- A **set of outputs** called the **range**.

We say that the output is a function of the input.

Example 9.2 (Do you understand the function concept?)
Which of the following describe functions?

1. Is the temperature in Lexington a function of the date? No (temp changes in a day)
2. Is the high temperature in Lexington a function of the date? Yes
3. Is the amount of sales tax that you owe a function of the value of the merchandise you purchase? No (depends on the state)
4. Is the final score a function of the game? Yes
5. Is the game a function of the final score? No (many games have the same final score)

Example 9.3 (Do you understand the function concept?)
Could the following table of values represent the values of a function? Explain.

Input	2	3	-1	0
Output	5	1	5	7

Yes. Every input has exactly one output.

Example 9.4 (Do you understand the function concept?)
Could the following table of values represent the values of a function? Explain.

Input	1	3	2	0
Output	2	-1	5	7

Yes (same reason as above)

Example 9.5 (Do you understand the function concept?)
Could the following table of values represent the values of a function? Explain.

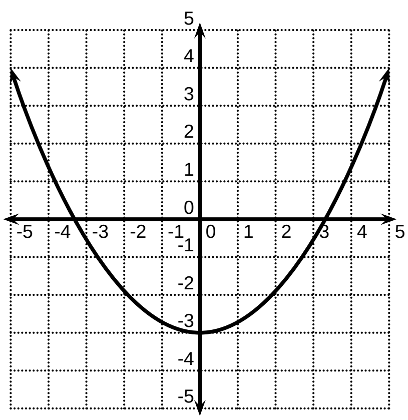
Input	1	3	2	1
Output	2	-1	5	7

No. The input 1 has two different outputs.

Example 9.6 (Do you understand the function concept?)

1. Does the graph below define y as a function of x? Yes
2. Does the graph below define x as a function of y? No

(for instance, if $y = 0$, there are two different x values)

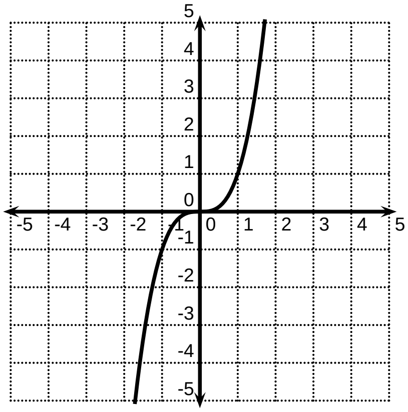


Note 9.7
Traditionally, x is the input variable and y is the output variable of many functions. Consequently, people often drop the variable names when discussing functions of this type. The following questions mean the same thing:

- Does the graph define y as a function of x?
- Does the graph define a function?

Example 9.8 (Do you understand the function concept?)

1. Does the graph below define a function? Yes
2. Does the graph below define x as a function of y? Yes



Example 9.9 (Do you understand the function concept?)
Does the equation $s^2 = 3t + 1$ define s as a function of t?

No. For the input $t=0$, we have $s^2=1$. This implies s could be ± 1 (two outputs for one input).

Example 9.10 (Do you understand the function concept?)
Does the equation $s^3 = 3t + 1$ define s as a function of t?

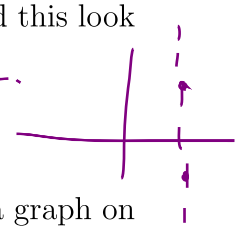
Yes.

Example 9.11 (Do you understand the function concept?)
Does the equation $s^2 = 3t + 1$ define t as a function of s?

Yes. (Note: $t = \frac{s^2-1}{3}$)

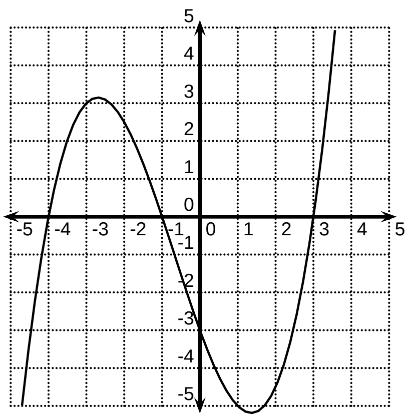
9.1.1 The Vertical Line Test

The definition of a function (with input x and output y) requires that for every x value, there is exactly one y value. So what would violate this requirement? What would this look like graphically? Two y-values for one x-value.



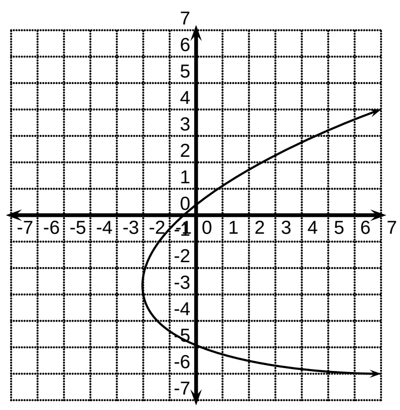
The **Vertical Line Test** is used to determine if y is a function of x. If you have a graph on the Cartesian Coordinate System and at least one vertical line touches the graph in more than one place, then y is NOT a function of x for that graph. If no vertical line touches the graph in more than one place, then y is a function of x for that graph.

Example 9.12 (Do you understand the vertical line test?)
Does the graph define a function?



Yes.

Example 9.13 (Do you understand the vertical line test?)
Is x a function of y in the graph below?



Yes

What test determines if x is a function of y?

Horizontal line test.

9.2 Function Notation

We have pictured a function as a machine that has one output for every input. Different machines have different purposes, so it makes sense to name the machines.

$$\text{INPUT} \longrightarrow \boxed{\text{NAME}} \longrightarrow \text{OUTPUT}$$

In this class, we will draw machine diagrams when they can help us to understand a concept. However, it can become quite tedious to draw these diagrams all of the time. Consequently, standard shorthand has been developed for describing functions. This shorthand is known as function notation.

Function Notation:

$$\text{name(input)} = \text{output}$$

Example 9.14 (Do you understand function notation?)
Let $\text{Jake}(x) = 3x - 6$.

- Draw the machine diagram that corresponds to this function.

$$x \longrightarrow \overset{\text{Jake}}{\boxed{}} \longrightarrow 3x-6$$

- What is $\text{Jake}(5)$?

$\text{Jake}(5) = 3(5) - 6 = 9$

- What is $\text{Jake}(\square)$?

$\text{Jake}(\boxed{}) = 3\boxed{} - 6$

- What is $\text{Jake}(3x-6)$?

$\text{Jake}(3x-6) = 3(3x-6) - 6$

Example 9.15 (Do you understand function notation?)
Let $f(x) = x^2 + 5$.

- What is $\dfrac{f(3) + f(2+y)}{f(a)}$?

$$\frac{f(3) + f(2+y)}{f(a)} = \frac{14 + (2+y)^2 + 5}{a^2 + 5}$$

$f(3) = 3^2 + 5 = 14$

$f(2+y) = (2+y)^2 + 5$

$f(a) = a^2 + 5$

- What is $\dfrac{f(x+h)-f(x)}{h}$?

$f(x+h) = (x+h)^2+5$
$= x^2+2xh+5$

$f(x) = x^2+5$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x^2+2xh+h^2+5)-(x^2+5)}{h}$$
$$= \frac{x^2+2xh+h^2+5-x^2-5}{h}$$
$$= \frac{2xh+h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}}$$
$$= 2x+h$$

(*NOTE:* This expression is known as a **difference quotient**. It plays a fundamental role in Calculus. Expect to see this often. Do not be surprised to see problems of this sort on homework, quizzes, and exams in this class.)

9.2.1 Piecewise-defined functions

Piecewise-defined functions are functions that use different formulas for different inputs. These functions are described by listing the acceptable inputs for each formula after the formula.

For example, consider

$$f(x) = \begin{cases} 7-x & \text{if } x \leq 2 \\ x^3 & \text{if } x > 2 \end{cases}.$$

This definition says that we will use the formula $7-x$ when $x \leq 2$, but we will use the formula x^3 when $x > 2$. Notice that every input value corresponds to exactly one of the formulas.

Example 9.16 (Do you understand piecewise-defined functions?)
Let

$$k(x) = \begin{cases} x^2 & \text{if } x < 5 \\ x+3 & \text{if } x \geq 5 \end{cases}$$

- Find $k(7)$. $k(7) = 7+3 = 10$

- Find $k(-1)$. $k(-1) = (-1)^2 = 1$

- Find $k(5)$. $k(5) = 5+3 = 8$

Example 9.17 (Do you understand piecewise-defined functions?)
Let

$$h(x) = \begin{cases} 2 & \text{if } x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

- Find $h(4)$. $h(4) = 0$
- Find $h(0)$. $h(0) = 2$
- Find $h(1)$. $h(1) = 2$
- Find $h(2)$. $h(2) = 0$

9.3 The Domain of a Function

Definition 9.18
The **domain** of a function is the set of all the values that are permitted as input values for the function.

If the domain is not specified, the domain is the set of all real number inputs which produce a real number output when processed by the function.

When determining the domain, you should watch for:

- numbers that will produce a zero in a denominator
- numbers that will produce a negative number inside an even root.

You do not want to include these numbers in the domain. Later in the semester, we will learn about other numbers that can cause problems for some special functions.

Example 9.19 (Do you understand domain?)
Find the domain of the following functions.

- $a(x) = x^7 + 3x - 7$

  $(-\infty, \infty)$

- $b(x) = \dfrac{2x+4}{x+2}$

  $x + 2 \neq 0$

  $x \neq -2$

-2

$(-\infty, -2) \cup (-2, \infty)$

- $c(x) = \sqrt{x+7}$ $x+7 \geq 0$ $[-7, \infty)$

  $-$ | $+$

  -7

- $d(x) = \sqrt[3]{x+7}$ $(-\infty, \infty)$

- $e(x) = \dfrac{1}{\sqrt{x+7}}$ $x+7 \geq 0$ AND $\sqrt{x+7} \neq 0$

  $[-7, \infty)$ $x+7 \neq 0$

  $x \neq -7$

  $(-7, \infty)$

9.4 The Graph of a Function

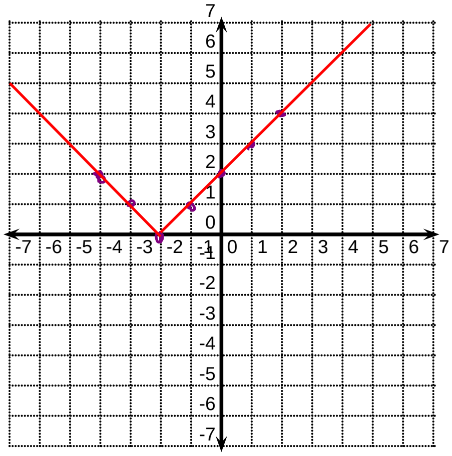
Definition 9.20
The graph of the function f is the set of all points $(x, f(x))$ where x is in the domain of f.

We can plot these points on an xy-Cartesian Coordinate System. You can see that the output values of the function (i.e., the $f(x)$ values) are the y-coordinates of the points.

You are responsible for graphs of basic functions.
You will need to know how to graph some basic functions without the help of your calculator.

- Linear Functions ($f(x) = mx + b$)
- Power Functions ($f(x) = x^n$) where n is a positive integer.
- Square Root Function ($f(x) = \sqrt{x}$)
- Greatest Integer Function ($f(x) = [\![x]\!]$)
- Absolute Value Function ($f(x) = |x|$)
- Piecewise-defined Functions.

Example 9.21
Sketch the graph of $g(x) = |x + 2|$.



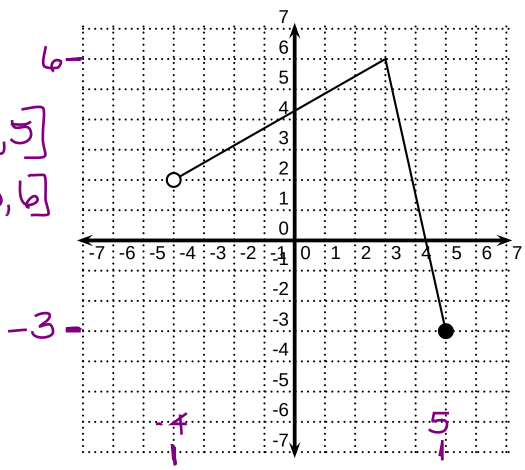
x	$g(x) = y$
0	2
1	3
2	4
-1	1
-2	0
-3	1
-4	2

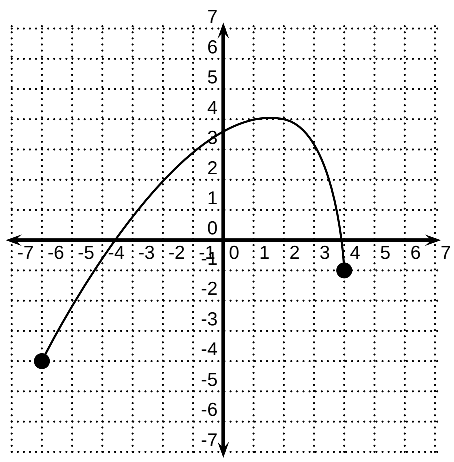
What is the domain of $g(x)$? What is the range?

D: $(-\infty, \infty)$
R: $[0, \infty)$

Example 9.22 (Domain and Range from a Graph)
Find the domain and range of each of the functions graphed below.

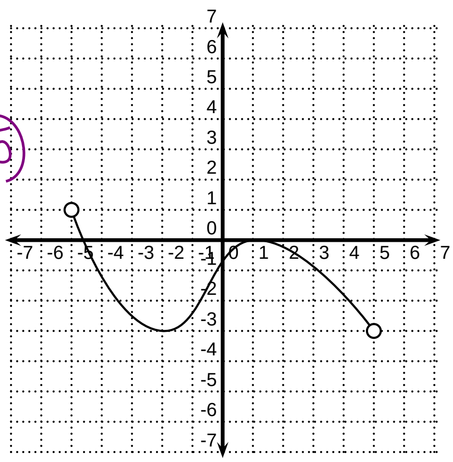
D: $(-4, 5]$
R: $[-3, 6]$

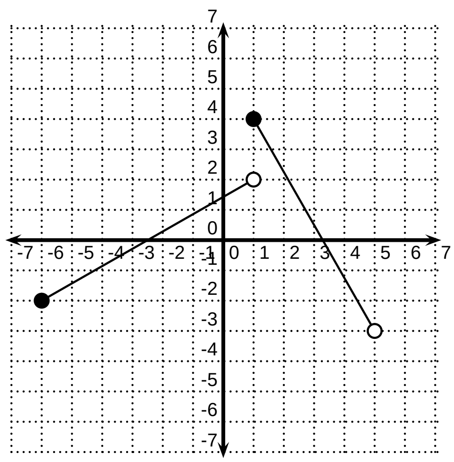




D: $[-6, 4]$
R: $[-4, 4]$

D: $(-5, 5)$
R: $[-3, 1)$

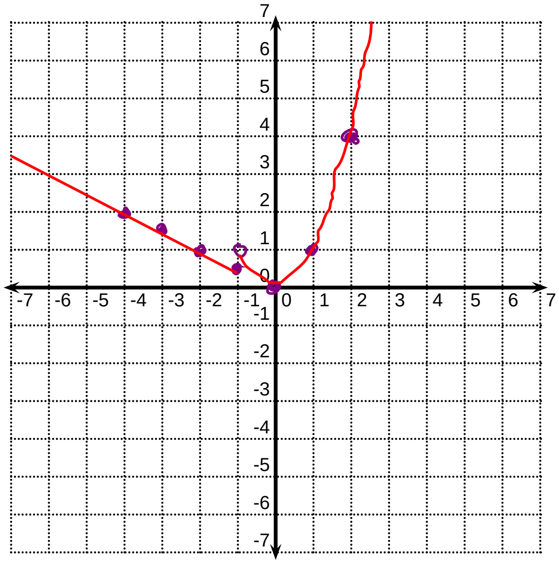




D: $[-6, 5)$
R: $(-3, 4]$

Example 9.23 (Graphing Piecewise-defined Functions)
Sketch the graph of

$$k(x) = \begin{cases} \frac{1}{2}x + 1 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$$



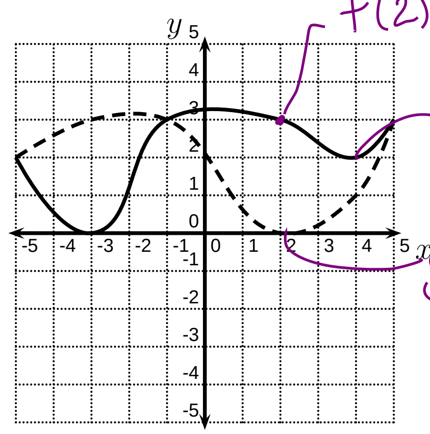
Two pieces

x	$\frac{1}{2}x+1$
-4	-1
-3	$-\frac{1}{2}$
-2	0
-1	$\frac{1}{2}$

x	x^2
-1	1
0	0
1	1
2	4

Example 9.24 (Can you interpret the graph of a function?)
In the picture below, the graph of $y = f(x)$ is the solid graph, and the graph of $y = g(x)$ is the dashed graph. Find the true statement.

$f(2)=3$



$f(4)=2$

$g(2)=0$

Possibilities:

~~(a)~~ $f(2) < g(2)$ $\quad 3 \not< 0$

~~(b)~~ $f(4) = 1$ $\quad f(4)=2$

(c) $f(-3) < g(-3)$ $\quad 0 < 3$ ✓

~~(d)~~ $f(-1) > g(-1)$ $\quad 3 \not> 3$

~~(e)~~ $f(-1) = 2$ $\quad f(-1) \neq 2$

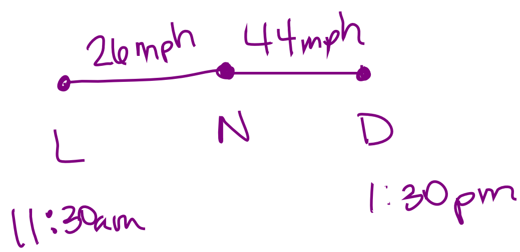
9.5 Average Rates of Change

We briefly introduced rates of change when we discussed the slope of a line. Recall that rates of change arise when two different quantities are changing simultaneously. The word "**per**" is very often associated with a rate of change. Think about the speed of a car. The position of the car is changing as the time changes. Both position and time are changing. We measure the speed (a.k.a. the rate of change of the distance traveled by the car with respect to time) in miles **per** hour. We can think of this as $\frac{\Delta d}{\Delta t}$. This looks a lot like the formula for the slope of a line.

Example 9.25 (Average Speed)
Joni leaves the house at 9:00AM. She travels 5 miles to the grocery store, stops the car, shops, and drives back home. She returns to her home at 11:30AM. What is the average speed of Joni's car from 9:00AM to 11:30AM?

$$\frac{\Delta d}{\Delta t} = \frac{10}{2.5} = 4\text{mph}$$

Example 9.26 (Average Speed)
Juliet leaves at 11:30AM from Lexington and arrives in Nicholasville at noon to stop and have lunch. At 1:00PM, Juliet leaves Nicholasville and heads to Danville. She arrives in Danville at 1:30PM. From Lexington to Nicholasville, Juliet's average speed is 26 miles per hour(mph) and from Nicholasville to Danville, Juliet's average speed is 44mph. What is her average speed from Lexington to Danville (including the time she took for lunch)?



Average Speed $L \to D$ $\frac{\Delta d}{\Delta t} = \frac{35}{2} = 17.5\text{mph}$

$L \to N$ dist $(0.5)(26) = 13\text{miles}$

$N \to D$ dist $(0.5)(44) = 22\text{miles}$

$L \to D$ dist $13 + 22 = 35\text{miles}$

To find the **Average Rate of Change** of one quantity with respect to another quantity, you simply divide the change in quantity 1 by the change in quantity 2.

$$\frac{\Delta \text{ quantity 1}}{\Delta \text{ quantity 2}}$$

Average speed as in the previous examples is one common introductory example of an average rate of change.

Definition 9.27 (Average Rate of Change of a Function)
If f is a function, then the **average rate of change of** $f(x)$ **with respect to** x **as** x **changes from** a **to** b is given by

$$\frac{f(b)-f(a)}{b-a} = \frac{f(a)-f(b)}{a-b} = \frac{\Delta f(x)}{\Delta x}$$

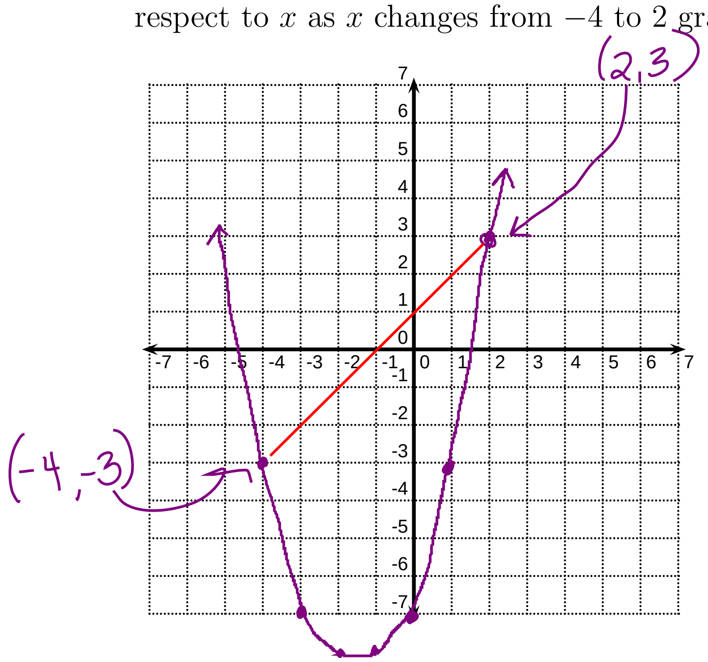
Example 9.28 (Average Rate of Change of a Function)
Let $g(x) = x^2 + 3x - 7$. Find the average rate of change of $g(x)$ with respect to x as x changes from -4 to 2.

$g(2) = 2^2 + 3(2) - 7 = 3$

$g(-4) = (-4)^2 + 3(-4) - 7 = -3$

$$\frac{g(2)-g(-4)}{2-(-4)} = \frac{3-(-3)}{6} = 1$$

Example 9.29 (Average Rate of Change of a Function)
Let $g(x) = x^2 + 3x - 7$. How can you interpret the average rate of change of $g(x)$ with respect to x as x changes from -4 to 2 graphically?



x	$g(x)$
-4	-3
-3	-7
-2	-9
-1	-9
0	-7
1	-3
2	3

avg. Rate of Change is the slope of the line connecting $(-4, g(-4))$ to $(2, g(2))$

We can now see that the average rate of change of a function with respect to x is simply the slope of the line that connects two points on the graph of the function. The line that connects these two points is called a **secant line**.

Example 9.30 (Average Rate of Change of a Function)
Air is being pumped into a tire. When the outer radius of the tire is R feet, the volume of the air in the tire is given by $V(R) = \pi(R^2 - 1)$ where V is measured in ft^3. What is the average rate of change of the volume of air in the tire with respect to the outer radius of the tire as the outer radius changes from 18 inches to 2 feet?

AROC from $R=1.5$ to $R=2$

$$\frac{V(2)-V(1.5)}{2-1.5} = \frac{3\pi - 1.25\pi}{0.5} = \frac{1.75\pi}{0.5} = 3.5\pi \text{ cubic feet per foot}$$

$V(2) = \pi(2^2-1) = 3\pi$

$V(1.5) = \pi(1.5^2-1) = 1.25\pi$

9.5.1 Difference Quotients

It is common to compute the average rate of change of a function for many intervals at once with a common end point and also for small Δx values. For example, you might need to compute the average rate of change for x between 4 and 4.1, or between 5 and 5.01, or between 7 and 6.999. Sometimes it is even necessary to compute several of these values. If this is the case, it is often helpful to find a general formula for the average rate of change.

If you are computing the average rate of change from x to $x + h$, what is Δx? $x+h-x = h$

Example 9.31 (Difference Quotient)
Let $g(x) = x^2 + 3x + 5$. Find the average rate of change of $g(x)$ on the interval from a to $a + h$. (You should assume that $h \neq 0$. Nothing changed if $h = 0$.)

$$\frac{g(a+h)-g(a)}{x+h-x} = \frac{g(a+h)-g(a)}{h} = \frac{(a^2+2ah+h^2+3a+3h+5)-(a^2+3a+5)}{h}$$

$g(a+h) = (a+h)^2 + 3(a+h) + 5$

$= a^2+2ah+h^2+3a+3h+5$

$g(a) = a^2+3a+5$

$$= \frac{a^2+2ah+h^2+3a+3h+5-a^2-3a-5}{h}$$

$$= \frac{2ah+h^2+3h}{h}$$

$$= \frac{h(2a+h+3)}{h} = 2a+h+3$$

Example 9.32 (Optional)
Use your result from the previous example to find the average rate of change of $g(x)$ on the interval from a to b. What is x for each problem? What is h?

a	b	x	h	Ave ROC
3	3.1	3	0.1	9.1
3	3.01	3	0.01	9.01
3	3.001	3	0.001	9.001
3	2.9	3	-0.1	8.9
3	2.99	3	-0.01	8.99
3	2.999	3	-0.001	8.999

$2(3)+0.1+3=9.1$

$2(3)+0.01+3=9.01$

Definition 9.33
Let f be a function. The **difference quotient** of f is the average rate of change of f on the interval from x to $x+h$.

$$\frac{f(x+h)-f(x)}{(x+h)-x}=\frac{f(x+h)-f(x)}{h}$$

Example 9.34 (Difference Quotient)
Let $d(x)=3x+17$. Find the average rate of change of $d(x)$ on the interval from x to $x+h$. (You should assume that $h\neq 0$. Nothing changed if $h=0$.)

$$\frac{(3(x+h)+17)-(3x+17)}{h}=\frac{3x+3h+17-3x-17}{h}=\frac{3h}{h}=3$$

What does the graph of $d(x)$ look like? Why could you have predicted this average rate of change without computation?

The graph of $d(x)$ is a line. The average rate of change is the slope of the line through $(a,f(a))$ and $(b,f(b))$ which is exactly the graph of $d(x)$.

9.6 Operations on Functions

There are five basic operations on functions. The first four operations are straightforward. We will examine function composition in a bit more detail.

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$

How do you determine the **domain** of $f+g$, $f-g$, fg, and $\dfrac{f}{g}$?

- x must be in the domain of f.
- x must be in the domain of g.
- In the domain of $\dfrac{f}{g}$, $g(x) \neq 0$.

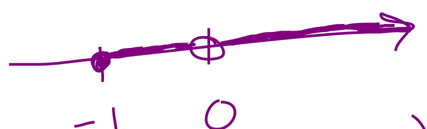
Example 9.35 (Operations on Functions Domain)
Let $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{x+1}$.

- Find $(g+f)(3)$. $(g+f)(3) = g(3) + f(3) = \sqrt{3+1} + \frac{1}{3} = 2 + \frac{1}{3} = \frac{7}{3}$
- Find $\frac{g}{f}(3)$. $\frac{g}{f}(3) = \dfrac{g(3)}{f(3)} = \dfrac{\sqrt{3+1}}{1/3} = 3 \cdot 2 = 6$
- Find $(fg)(x)$.

$(fg)(x) = f(x) \cdot g(x) = \frac{1}{x}\sqrt{x+1}$

- Find the domain of $(fg)(x)$.

$$\left.\begin{array}{l} \text{Domain of } f: x \neq 0 \\ \text{Domain } g: x+1 \geq 0 \rightarrow x \geq -1 \end{array}\right\}$$



$[-1, 0) \cup (0, \infty)$

- Find the domain of $\frac{g}{f}(x)$.

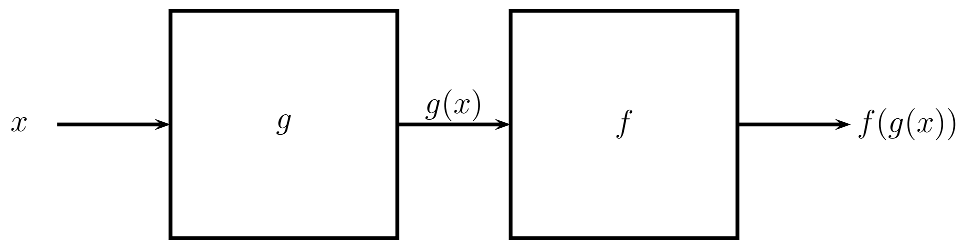
$$\left.\begin{array}{l} \text{Domain of } f: x \neq 0 \\ \text{Domain of } g: x \geq -1 \end{array}\right\} \quad [-1, 0) \cup (0, \infty)$$

$f(x) \neq 0$

$\frac{1}{x} \neq 0$

No Sol'ns

The machine diagram for the function composition $f \circ g(x) = f(g(x))$ is shown below.



How do you determine the **domain** of $f(g(x))$?

- x must be in the domain of g.
- $g(x)$ must be in the domain of f.

Example 9.36 (Function Composition and Domain)

Let $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{x+1}$.

- Find $f(g(8))$. $= f\left(\sqrt{8+1}\right) = f(3) = \frac{1}{3}$
- Find $f(g(x))$. $f(g(x)) = f\left(\sqrt{x+1}\right) = \dfrac{1}{\sqrt{x+1}}$
- Find the domain of $f(g(x))$.

  Domain g: input ≥ -1 → $x \geq -1$

  Domain f: input $\neq 0$ → $\sqrt{x+1} \neq 0 \rightarrow x \neq -1$

  $\boxed{(-1, \infty)}$
- Find $g(f(x))$.

  $g\left(\frac{1}{x}\right) = \sqrt{\frac{1}{x} + 1}$
- Find $f(f(x))$.

  $f(f(x)) = f\left(\frac{1}{x}\right) = \dfrac{1}{\frac{1}{x}} = 1 \cdot \dfrac{x}{1} = x$
- Find the domain of $f(f(x))$.

  Domain of f: input $\neq 0$ → $x \neq 0$

  Domain of f: input $\neq 0$ → $\frac{1}{x} \neq 0$ (No Sol'n)

  $x \neq 0$

  $(-\infty, 0) \cup (0, \infty)$

Example 9.37 (Do You Understand Function Compostion?)
Let $f(x) = 5x + 2$ and $g(x) = \dfrac{x-2}{5}$.

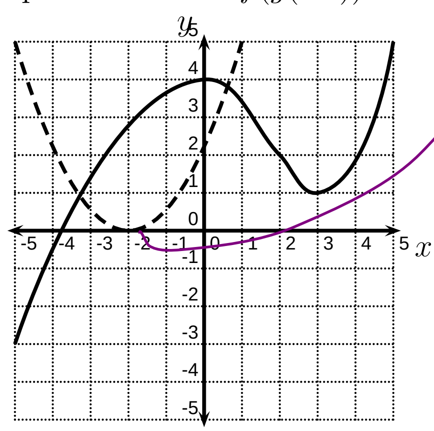
- Find $f(g(7))$. $f\left(\frac{7-2}{5}\right) = f(1) = 5 \cdot 1 + 2 = 7$

- Find $f(g(x))$. $f\left(\frac{x-2}{5}\right) = 5 \cdot \left(\frac{x-2}{5}\right) + 2 = x - 2 + 2 = x$

- Find $g(f(x))$. $g(5x+2) = \dfrac{5x+2-2}{5} = \dfrac{5x}{5} = x$

Example 9.38 (Do You Understand Function Compostion?)
In the picture below, the graph of $y = f(x)$ is the solid graph, and the graph of $y = g(x)$ is the dashed graph. Use the graphs to evaluate $f(g(-2))$.

Possibilities:



$g(-2) = 0$

$f(g(-2)) = f(0) = 4$

(a) 0

(b) 2

(c) 4

(d) -3

(e) 1

Example 9.39
Suppose $f(x) = |x|$ and $g(x) = |\underbrace{2x+1}_{\text{input of abs. value}}| - 5$. Write $g(x)$ in terms of $f(x)$.

$g(x) = f(2x+1)$

Example 9.40 (An Application of Function Composition)
A rock is dropped into a pond creating circular ripples. The radius of the outer circle is increasing at a rate of 3 feet per second. Express the area enclosed by the outer circle as a function of time.

$$A = \pi r^2 = \pi(3t)^2 = 9t^2\pi$$

- Why is this question in a section about function composition?

To find the area of the outer circle, you must first find the radius as a function of time and then use the area formula for a circle.

- Express the radius as a function of time.

$$r(t) = 3t$$

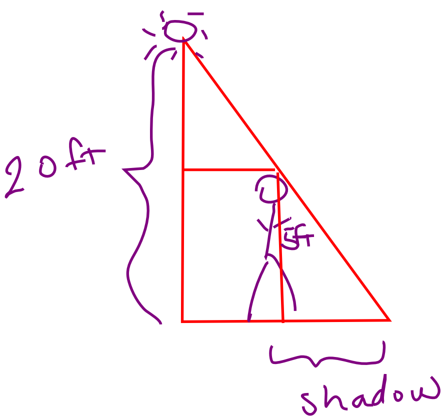
- Express the area as a function of the radius.

$$A(r) = \pi r^2$$

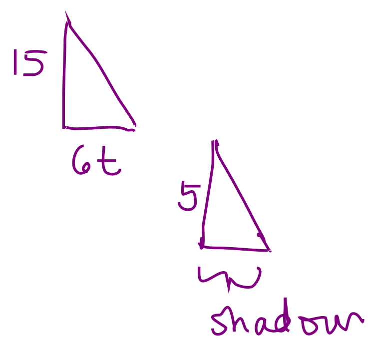
- How do these functions relate to the first function you found in this example?

$$A(t) = A(r(t)) = A(3t) = \pi(3t)^2 = 9\pi t^2$$

Example 9.41 (An Application of Function Composition)
Joni is leaning against a lamppost that is 20 feet high. She begins walking away from the lamppost. Joni is 5 feet tall and she is walking at a rate of 6 feet per second. Express the length of Joni's shadow as a function of time.



Similar Δs



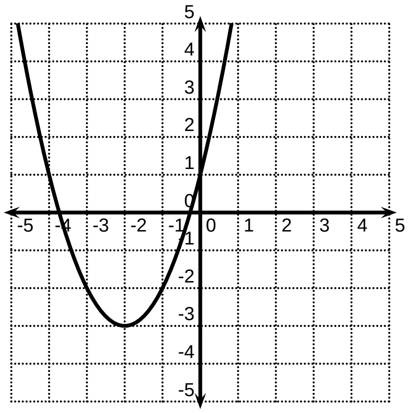
Let t be the # of seconds since Joni started walking.

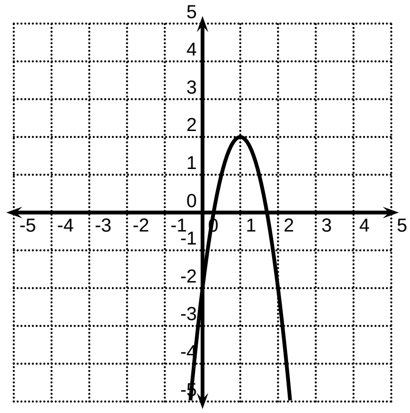
$$\frac{\text{shadow}}{5} = \frac{6t}{15}$$

$$\text{shadow} = \frac{6t}{15} \cdot 5 = 2t$$

9.7 Graph Transformations

The graphs shown below have the same basic shape, but they are different. How?





These graphs have the same basic shape, but they have been transformed by shifts, scaling factors, and reflections. In this section, we will learn how simple modifications of a function by a constant can change the graph of the function. We first consider how a simple modification affects one point on the graph of a function.

Example 9.42 (Do you understand graph transformations?)
Suppose that the graph of f contains the point $(2, 3)$. Find a point that must be on the graph of g. Explain how you had to move the point on the original graph f to obtain a point on the new graph g.

$\boxed{f(2) = 3}$

- The graph of $g(x) = f(x) + 5$ must contain the point $(2, 8)$. Shift up 5 units.

  $g(2) = f(2) + 5 = 3 + 5 = 8$

  $g(2) = 8$

- The graph of $g(x) = f(x) - 5$ must contain the point $(2, -2)$. Shift down 5 units

  $g(2) = f(2) - 5 = 3 - 5 = -2$

  $g(2) = -2$

- The graph of $g(x) = f(x + 5)$ must contain the point $(-3, 3)$. Shift left 5 units.

  $x + 5 = 2$

  $x = -3$

  $g(-3) = f(-3 + 5) = f(2) = 3$

  $g(-3) = 3$

- The graph of $g(x) = f(x-5)$ must contain the point $(7,3)$.

$x-5=2$

$x=7$

$g(7)=f(7-5)=f(2)=3$

$g(7)=3$

Shift Right 5 units.

- The graph of $g(x) = 5f(x)$ must contain the point $(2,15)$.

$g(2)=5f(2)=5\cdot 3=15$

$g(2)=15$

Scale vertically by 5.

- The graph of $g(x) = \frac{1}{5}f(x)$ must contain the point $(2,\frac{3}{5})$.

$g(2)=\frac{1}{5}f(2)=\frac{1}{5}\cdot 3=\frac{3}{5}$

$g(2)=\frac{3}{5}$

Scale vertically by $\frac{1}{5}$.

- The graph of $g(x) = f(5x)$ must contain the point $(\frac{2}{5},3)$.

$5x=2$

$x=\frac{2}{5}$

$g(\frac{2}{5})=f(5\cdot\frac{2}{5})=f(2)=3$

Scale horizontally by $\frac{1}{5}$.

- The graph of $g(x) = f\left(\frac{1}{5}x\right)$ must contain the point $(10,3)$.

$\frac{1}{5}x=2$

$x=10$

$g(10)=f(\frac{1}{5}\cdot 10)=f(2)=3$

$g(10)=3$

Scale Horiz. by 5.

- The graph of $g(x) = f(7x)+5$ must contain the point $(\frac{2}{7},8)$.

$7x=2$

$x=2/7$

$g(2/7)=f(7\cdot\frac{2}{7})+5=f(2)+5=3+5=8$

- **A Challenge:** The graph of $g(x) = 3f(7x+1)+5$ must contain the point $(\frac{1}{7},14)$.

$7x+1=2$

$7x=1$

$x=\frac{1}{7}$

$g(\frac{1}{7})=3f(7\cdot\frac{1}{7}+1)+5$

$=3f(2)+5$

$=3\cdot 3+5$

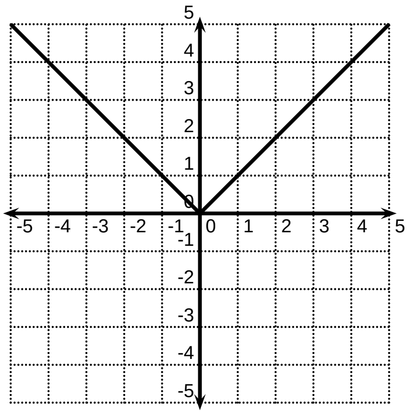
$=14$

$g(\frac{1}{7})=14$

Suppose that the graph of f contains the point (a, b). The chart below lists several new functions g which are transformations of f, a point which must be on each transformed graph, and descriptions of the graphical transformations.

New Function	New Point	Transformation	New Function	New Point	Transformation
$g(x) = f(x) + d$	$(a, b + d)$	Shift up d units	$g(x) = f(x + c)$	$(a - c, b)$	Shift left c units
$g(x) = f(x) - d$	$(a, b - d)$	Shift down d units	$g(x) = f(x - c)$	$(a + c, b)$	Shift right c units
$g(x) = -f(x)$	$(a, -b)$	Reflect about x-axis	$g(x) = f(-x)$	$(-a, b)$	Reflect about y-axis
$g(x) = df(x)$	(a, db)	Scale Vertically by a factor of d	$g(x) = f(cx)$	$\left(\frac{a}{c}, b\right)$	Scale Horizontally by a factor of $\frac{1}{c}$
$g(x) = \frac{1}{d}f(x)$	$\left(a, \frac{b}{d}\right)$	Scale Vertically by a factor of $\frac{1}{d}$	$g(x) = f\left(\frac{x}{c}\right)$	(ac, b)	Scale Horizontally by a factor of c

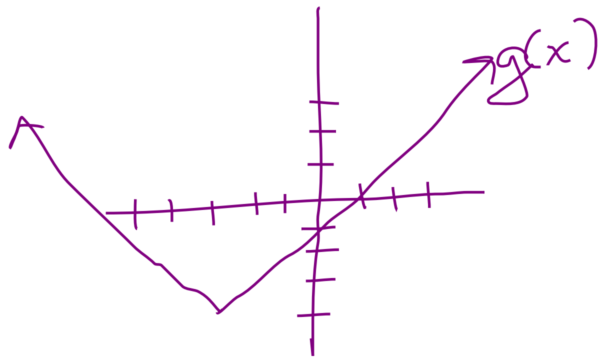
Example 9.43 (Do you understand graph transformations?)
Let $f(x) = |x|$. Write $g(x)$ in terms of $f(x)$ and explain how you would transform the graph of f to draw the graph of g. Sketch the graph of g.



- $g(x) = |x + 3| - 4$.

$g(x) = f(x+3) - 4$

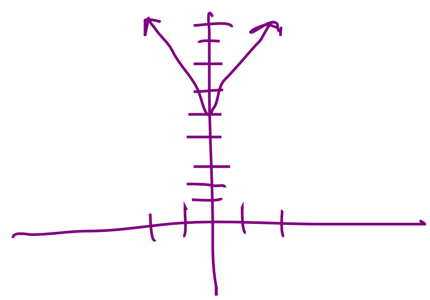
- shift left 3
- shift down 4



- $g(x) = 2|x| + 5$.

$g(x) = 2f(x) + 5$

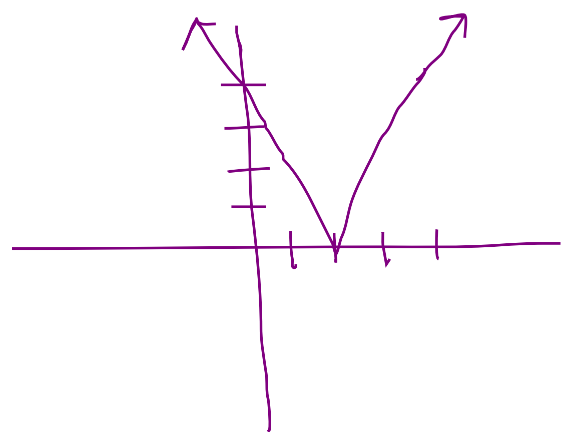
- scale vert by 2
- shift up 5



- $g(x) = |2x - 4|$.

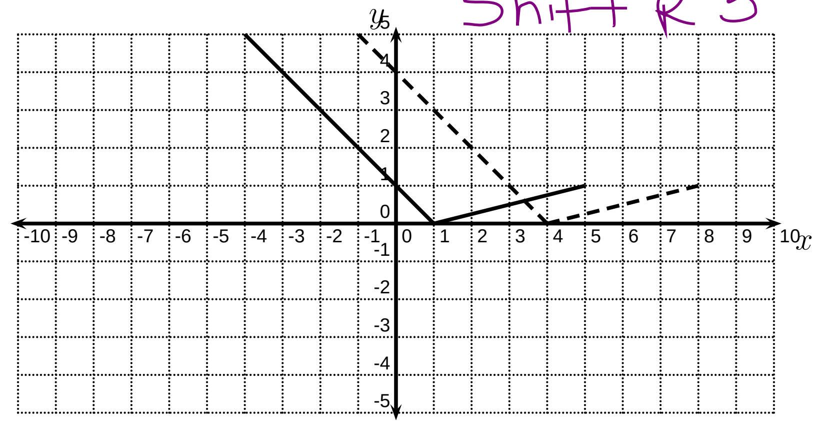
$g(x) = f(2x-4)$

- shift right 4
- scale horiz $\frac{1}{2}$



Example 9.44 (Do you understand graph transformations?)
In the picture below, the graph of $y = f(x)$ is the solid graph, and the graph of $y = g(x)$ is the dashed graph. Find a formula for $g(x)$.

Shift R 3



Possibilities:

(a) $g(x) = f(x-3)$

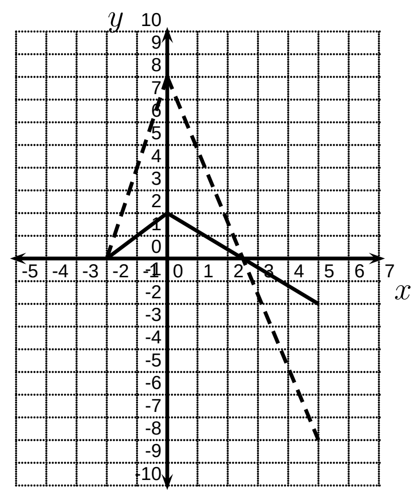
(b) $g(x) = f(x) - 3$

(c) $g(x) = -3f(x)$

(d) $g(x) = f(x) + 3$

(e) $g(x) = f(x+3)$

Example 9.45 (Do you understand graph transformations?)
In the picture below, the graph of $y = f(x)$ is the solid graph, and the graph of $y = g(x)$ is the dashed graph. Find a formula for $g(x)$.



Scale vertically by 4

Possibilities:

(a) $g(x) = -\frac{1}{4}f(x)$

(b) $g(x) = -4f(x)$

(c) $g(x) = \frac{1}{4}f(x)$

(d) $g(x) = 4f(x)$

(e) $g(x) = f(4x)$

9.8 One-to-one Functions and Inverse Functions

9.8.1 One-to-one Functions (1-1 Functions)

Recall that each input of a function can only correspond to one output. Now suppose that we have the output and wish to know what the input of the function was. When is this possible?

Example 9.46 (One-to-one Functions)
Let T be the function that maps dates to the high temperature in Lexington, Kentucky on that date. Suppose that I tell you that the $T(date) = 81°$. Is it possible for you to know which date was input into T?

No, more than 1 date can have a high temp of 81°.

Example 9.47 (One-to-one Functions)
Let S be the function that maps an American citizen to his or her Social Security Number. Suppose that $S(person) = 111 - 11 - 1111$. Assuming that this is a valid Social Security Number and that you had all the resources of the FBI at your fingertips, could you tell me what person was input into S?

Yes.

One-to-one functions are functions that allow you to go backwards from the output to the input. In order to do this, you need to be sure that no output ever came from two different inputs.

Definition 9.48 (One-to-one Functions)
A function f is **one-to-one** if $f(a) = f(b)$ implies that $a = b$.

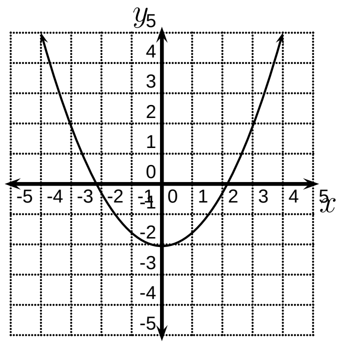
This definition says that whenever the outputs are the same, the inputs must be the same also. Equivalently, if the inputs are different, the outputs must be different also.

9.8.2 Graphs of One-To-One Functions

To determine if a graph corresponds to a one-to-one function, we need to check two things.

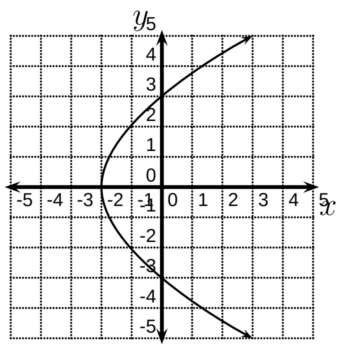
- Each input must correspond to exactly one output. In other words, the graph must first represent a function. (Vertical Line Test)
- Each output must correspond to exactly one input. This is a test to make sure that the function is one-to-one. For this, we need to use the Horizontal line test

Example 9.49 (Horizontal Line Test)
Assuming that x is the input and y is the output, does the graph below represent a one-to-one function?



No

Example 9.50 (Horizontal Line Test)
Assuming that x is the input and y is the output, does the graph below represent a one-to-one function?

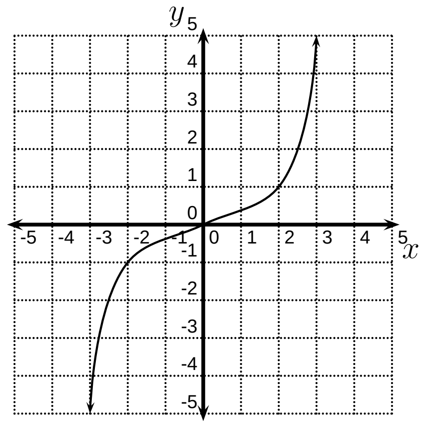


No

Why do you believe this example was included in the notes?

You must start w/ a function!

Example 9.51 (Horizontal Line Test)
Assuming that x is the input and y is the output, does the graph below represent a one-to-one function?



yes

9.8.3 Inverse Functions

If a function, f, is one-to-one, then there is another function g that, in some sense, is the reverse function for f. The input values of g are the output values from f, and the outputs values for g are the input values of f. This whole thing is a bit topsy-turvy. In essence, g is the function that you obtain when you hit the reverse button on the f function machine. Traditionally, g is called the inverse function of f and is denoted $g = f^{-1}$. Notice that the -1 in the notation is just that, notation. It **DOES NOT** mean that you take f to the -1 power. It can sometimes be confusing notation for students, but it is in all the literature so we must deal with it.

Definition 9.52 (Inverse Functions)
If f is a one-to-one function, then the **inverse function of** f, denoted f^{-1}, is the function defined by the following rule:

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

What is the domain of f^{-1}?

Range of f

What is the range of f^{-1}?

Domain of f

Example 9.53 (Inverse Functions)
Let $f(x) = \dfrac{2x+7}{5}$. Find $f^{-1}(3)$.

$$\boxed{f^{-1}(3) = 4}$$

$$3 = \frac{2x+7}{5}$$
$$15 = 2x+7$$
$$8 = 2x$$
$$4 = x$$

The key to finding a formula for f^{-1}, assuming that one exists, is to remember that x and y switch roles between f and f^{-1}.

Example 9.54 (Inverse Functions)
Let $f(x) = \dfrac{2x+7}{5}$. If f has an inverse function, find a formula for $f^{-1}(x)$. If f does not have an inverse function, can you find a way to restrict the domain of f so that it does have an inverse function.

$f: y = \dfrac{2x+7}{5}$

$f^{-1}: x = \dfrac{2y+7}{5}$

Solve for y.

$$x = \frac{2y+7}{5}$$
$$5x = 2y+7$$
$$5x-7 = 2y$$
$$\frac{5x-7}{2} = y$$
$$\boxed{f^{-1}(x) = \frac{5x-7}{2}}$$

Example 9.55 (Inverse Functions)
Let $g(x) = 5x^3 - 7$. If g has an inverse function, find a formula for $g^{-1}(x)$. If g does not have an inverse function, can you find a way to restrict the domain of g so that it does have an inverse function.

$g: y = 5x^3 - 7$

$g^{-1}: x = 5y^3 - 7$

Solve for y.

$$x = 5y^3 - 7$$
$$x+7 = 5y^3$$
$$\frac{x+7}{5} = y^3$$
$$\sqrt[3]{\frac{x+7}{5}} = y$$
$$g^{-1}(x) = \sqrt[3]{\frac{x+7}{5}}$$

Example 9.56 (Inverse Functions)
Let $h(x) = \dfrac{x+1}{2x+3}$. If h has an inverse function, find a formula for $h^{-1}(x)$. If h does not have an inverse function, can you find a way to restrict the domain of h so that it does have an inverse function.

$h: y = \dfrac{x+1}{2x+3}$

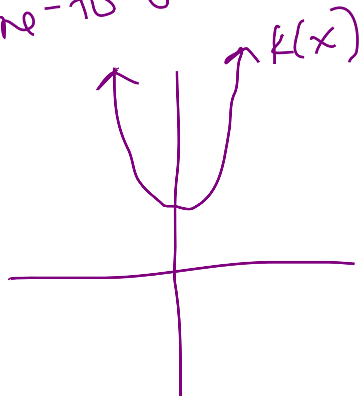
$h^{-1}: x = \dfrac{y+1}{2y+3}$

Solve for y.

$$x = \frac{y+1}{2y+3}$$
$$x(2y+3) = y+1$$
$$2xy+3x = y+1$$
$$3x-1 = y-2xy$$
$$3x-1 = y(1-2x)$$
$$\frac{3x-1}{1-2x} = y$$
$$\boxed{h^{-1}(x) = \frac{3x-1}{1-2x}}$$

Example 9.57 (Inverse Functions)
Let $k(x) = x^2 + 4$. If k has an inverse function, find a formula for $k^{-1}(x)$. If k does not have an inverse function, can you find a way to restrict the domain of k so that it does have an inverse function.

Not one-to-one!



Can make $k(x)$ one-to-one if we restrict the domain to $[0, \infty)$ (Could have also used $(-\infty, 0]$)

$y = x^2 + 4$

k^{-1}: $x = y^2 + 4$

$x - 4 = y^2$

$\sqrt{x-4} = y$ (b/c domain of $k(x)$ is nonneg)

$\boxed{k^{-1}(x) = \sqrt{x-4}}$

Theorem 9.58 (The Round Trip Theorem)
The functions f and g satisfy both the following properties if and only if they are one-to-one functions and inverses of each other. (In other words $f^{-1} = g$ and $g^{-1} = f$.)

- *$f(g(x)) = x$ for all x in the domain of g*
- *$g(f(x)) = x$ for all x in the domain of f*

Example 9.59 (The Round Trip Theorem)
Verify that f and f^{-1} from Example 9.54 satisfy the properties in the Round Trip Theorem.

$f(x) = 2x + 7$

$f^{-1}(x) = \dfrac{x-7}{2}$

$f(f^{-1}(x)) = f\left(\dfrac{x-7}{2}\right) = 2\left(\dfrac{x-7}{2}\right) + 7 = x - 7 + 7 = x$

$f^{-1}(f(x)) = f^{-1}(2x+7) = \dfrac{2x+7-7}{2} = \dfrac{2x}{2} = x$

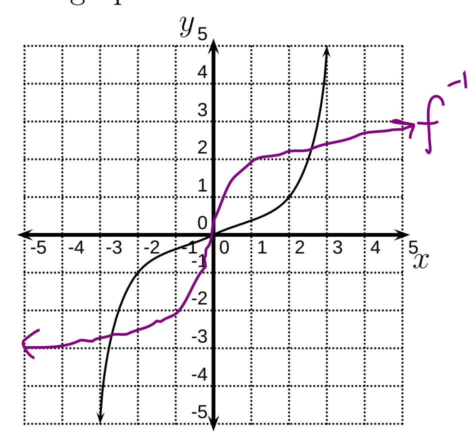
Round trip Theorem verified.

9.8.4 Graphs of Inverse Functions

Property 9.60 (Graphs of Inverse Functions)
If f is a one-to-one function and (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}.

Property 9.61 (Graphs of Inverse Functions)
If f is a one-to-one function, then the graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$.

Example 9.62 (Graphs of Inverses Functions)
The graph of the one-to-one function f is shown below. Sketch the graph of f^{-1}.



f	f^{-1}
$(0,0)$	$(0,0)$
$(2,1)$	$(1,2)$
$(3,5)$	$(5,3)$
$(-2,-1)$	$(-1,-2)$
$(-3,-5)$	$(-5,-3)$